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STOCKS AND FLOWS IN STATIC EQUILIBRIUM ANALYSIS

BY TATSURO ICHIISHI

I. INTRODUCTION

If we confine our attention to the *existence* of a general economic equilibrium and if, moreover, we are satisfied with *static* analysis, it could be argued that the analysis of general equilibrium systems has virtually been finished. The most complete account of this branch of theory might be found in Debreu's work, see [4]. With this analysis as the essential background, some economists considered factors that had been neglected in the pure models referred to above. Radner [11], for example, following Arrow [1], extended the model in an elegant way in order to give insights into the workings of an economy of an uncertain world. Malinvaud defined intertemporal efficiency in his model [7], [8], and explained the relationship between competitive equilibrium and intertemporal efficient allocation.

However, little attention is paid in these works to the distinction between stocks and flows of commodities. A reasonable interpretation of the models is therefore that no transactions in durable goods may occur; though economic units may buy and sell their services in addition to perishable commodities. But, in the economies that we actually observe, durable goods *are* transacted, and second-hand markets for them exist, although they are very imperfect and inadequate ones.

The purpose of this paper is to explore how far traditional static methods may be applied to the factors that are essential to an economy which embodies the variable, time. In particular, we shall modify the traditional model in order to introduce into the analysis the sale and purchase of durable goods, as well as of services.

By focusing attention on "stocks and flows", I hope to make the model a stepping-stone to dynamic models that contain money and bonds as well. Indeed it has often been noted that money and bonds, which appear essentially in dynamic economics, must in an uncertain world be regarded as a type of durable goods. (See, e.g., my survey article [6].)

The contents of this paper are divided into two parts: The first part, from Section II to Section VI, deals with a special model, in order to highlight its special features and characteristics. The nature of a stock, that its amount is nonnegative and that one cannot obtain an infinite amount of goods by saving finite amounts previously, will be shown to be part of the sufficient conditions which ensure the existence of a quasi-equilibrium. In the second part, Section VII, a general case will be discussed, in which the factors of uncertainty and depreciation are fully considered. It will be shown that, when analyzing stocks and flows of goods, Debreu's definition of events in [3] Ch. 7 is more useful than that in Radner [11].

The following mathematical notations will be employed throughout the paper:

R^n : n-dimensional Euclidian space.
Ω^n : nonnegative orthant of R^n.
$\| \ \|$: norm in R^n.
0 : zero element.

For any set X,

AX : asymptotic cone of X with vertex 0.
$\{\xi(t)\}_1^T$: $(\xi(1), \xi(2), \ldots, \xi(T))$.

II. BASIC CONCEPTS (I)

Commodities, $h = 1, 2, \ldots, l$:

We shall suppose the economy to have l kinds of commodities, where commodities are to be interpreted broadly, including perishable as well as durable goods and services, distinguished possibly according to location.

Future Market and Prices:

The economy is supposed to have T periods. Spot contracts for the first period, and future contracts for the subsequent periods up to the Tth are made at the beginning of the first period. A vector $p(1)$ of spot prices and vectors $p(2), \ldots, p(T)$ of prices for future contracts are determined in such a way that total demands and total supplies balance, where, by the hth coordinate $p_h(t)$ of $p(t) \in R^l$, we denote the future (or spot, if $t = 1$) price of the hth commodity of the tth period. It is also assumed that pure competition prevails in all the markets.

Producers, $j = 1, 2, \ldots, n$:

It is assumed that there are n agents called producers in the economy: Each producer determines, at the beginning of the first period, the production plan for the tth period when the jth producer uses commodities $a_j(t) \in R^l$ to produce commodities $b_j(t + 1) \in R^l$ at the next period ($t = 1, 2, \ldots, T$). Here, each hth coordinate of the vectors $a_j(t)$, $b_j(t + 1)$ denotes the stock amount of the hth commodity.

The relation between the input vector $a_j(t)$ and the output vector $b_j(t + 1)$ depends upon the level of technology, the feasible set of which will be written as $\mathscr{H}_{tj}$, i.e., $(a_j(t), b_j(t + 1)) \in \mathscr{H}_{tj}$ if and only if such a production process is technologically feasible. Technological *progress* of the disembodied type can be explained by the relation

$$\mathscr{H}_{1j} \subset \mathscr{H}_{2j} \subset \cdots \subset \mathscr{H}_{Tj},$$

although we do not necessarily assume this throughout this paper.

The content of inputs and outputs in this context should be understood in the same way as in von Neumann [9]. Outputs include intermediate products and remaining capital goods in addition to production goods. The diminishing-balance method of depreciation might well be explained by the shape of $\mathscr{H}_{tj}$ combined with the assumption of constant returns to scale, although we do not necessarily assume this in any context: If the hth commodity is a capital good, then

$$a_{jh}(t) - b_{jh}(t+1)$$

is the amount of its physical depreciation.(1)

Each producer is supposed to behave in such a way as to maximize his profit over T periods, subject to technological constraints, given spot prices and future prices. More specifically, the jth producer determines $\{a_j(t), b_j(t+1)\}_1^T$,

to maximize $\sum_{t=1}^{T} p(t)(b_j(t) - a_j(t))$,

subject to $(a_j(t), b_j(t+1)) \in \mathscr{H}_{tj}$, $t = 1, \ldots, T$,

given $b_j(1) = 0$ and $\{p(t)\}_1^T$.

Consumers, $i = 1, 2, \ldots, m$:

It is assumed that there are m agents called consumers in the economy: Each consumer determines, at the beginning of the first period, his consumption-saving-expenditure plan for the tth period when the ith consumer demands commodities as much as $x_i(t) + v_i(t+1) \in R^l$, so that he can then consume $x_i(t)$ units of commodities and save $v_i(t+1)$ units for the next period ($t = 1, \ldots, T$).

The meaning of the amount $x_i(t)$ of consumption flow, and the relation between $x_i(t)$ and $v_i(t+1)$ should be noted: If the hth commodity is a durable good, $x_{ih}(t)$ means the amount of physical depreciation due to the fact that the ith consumer uses the good at the tth period as much as $x_{ih}(t) + v_{ih}(t+1)$, while, if it is a perishable good or service, $v_{ih}(t+1) = 0$. We thus have completely durable goods (i.e., $x_{ih}(t) = 0$) on the one hand, completely perishable goods or services (i.e., $v_{ih}(t+1) = 0$) on the other hand, and commodities in various degrees between the two poles, the degrees being explained below

(1) Our model in this context is inferior to von Neumann's growth model [9], because we do not distinguish commodities according to age, so that only a special type of depreciation can be treated. The case which is no less general than that in von Neumann [9] will be treated in Section VII.

by the shape of the consumption set $\mathscr{C}_i$.

Define, for convenience,

$$\Delta v_i(t) \equiv v_i(t+1) - v_i(t) ,$$

and his problem is to choose

$$c_i \equiv \{x_i(t), \Delta v_i(t)\}_1^T \in R^{2lT} .$$

By the subset $\mathscr{C}_i$ of R^{2lT}, we denote the set of all such choices which are consistent with the physical nature of commodities and that of the ith consumer. The general condition $v_i(t) \geqq 0$, for example, must be imposed on the set $\mathscr{C}_i$, so that if $c_i \in \mathscr{C}_i$, then $\sum_{\tau=1}^{t} \Delta v_i(\tau) + v_i(1) \geqq 0$ for $t = 1, \ldots, T$. But some components of $x_i(t)$ may be negative, since the amount of services supplied by a consumer is measured in terms of a negative amount, as usual.

These considerations about $\mathscr{C}_i$ will rationalize the condition (a.1) shown in the next section.

Each consumer is assumed to have preference preordering on $\mathscr{C}_i$, which we denote by $\succsim_i$. $c_i \succsim_i c_i'$ means that the ith consumer prefers c_i to c_i', or is indifferent between the two.

Now, let $\omega_i(t) \in R^l$ be a commodity vector that "Nature" gives to the ith consumer at the tth period, and let θ_{ij} be the ratio of the profit of the jth producer which is a part of the wealth of the ith consumer. Each consumer is supposed to behave in such a way as to maximize his *utility* subject to his wealth constraint, given spot prices and future prices. More specifically, the ith cosumer determines $c_i \equiv \{x_i(t), \Delta v_i(t)\}_1^T$,

to maximize his preference,

subject to $c_i \in \mathscr{C}_i$,

$$\sum_{t=1}^{T} p(t)(x_i(t) + \Delta v_i(t)) \leqq \sum_{t=1}^{T} p(t)\Big(\omega_i(t) + \sum_{j=1}^{n} \theta_{ij}(b_j(t) - a_j(t))\Big) ,$$

given $p(t)$, $\omega_i(t)$, $a_j(t)$, $b_j(t)$, $t = 1, \ldots, T$, $j = 1, \ldots, n$, and $v_i(1)$.

Finally, when we consider the role of $\omega_i(1)$, we see that we can let, without loss of generality,

$$v_i(1) = 0 \quad \text{for any } i.$$

Also, since we are considering a private ownership economy, we shall assume,

$$b_j(1) = 0 \qquad \text{for} \quad j = 1, \ldots, n .$$

III. BASIC CONCEPTS (II)

In this section we will formulate some purely mathematical concepts, the economic interpretation of which is straightforward to any reader who has read the preceding section.

Production Set, $\mathscr{H}_{tj}$, $t = 1, 2, \ldots, T$, $j = 1, 2, \ldots, n$:

We are interested in a subset $\mathscr{H}_{tj}$ of R^{2l}, the element of which is written as $(a_j(t), b_j(t+1))$, where both $a_j(t)$ and $b_j(t+1)$ are points in R^l. Define:

$$\mathscr{Y}_j \equiv \{\{a_j(t), b_j(t+1)\}_1^T \in R^{2lT} | (a_j(t), b_j(t+1)) \in \mathscr{H}_{tj}, \quad t = 1, \ldots, T\},$$

$$\mathscr{H}_t \equiv \sum_{j=1}^{n} \mathscr{H}_{tj}, \qquad \mathscr{Y} \equiv \sum_{j=1}^{n} \mathscr{Y}_j.$$

We shall assume for any t and j,

(d.1) $\mathscr{H}_t \subset \Omega^{2l}$,

(d.2) $\mathscr{H}_t$ is convex and closed in R^{2l},

(d.3) $(0, b(t+1)) \in \mathscr{H}_t \Rightarrow b(t+1) = 0$,

(d.4) $A\mathscr{Y}$ does not degenerate to 0,

(d.5) $\mathscr{H}_{tj} \ni 0$.

Consumption Set, $\mathscr{C}_i$, $i = 1, 2, \ldots, m$:

Let $\mathscr{C}_i$ be a subset of $R^{lT} \times V$, where, if we define $\Delta v(t) \equiv v(t+1) - v(t)$ for any t,

$$V \equiv \left\{\{\Delta v(t)\}_1^T \in R^{lT} \middle| \sum_{\tau=1}^{t} \Delta v(\tau) \geqq 0, \quad \text{for any } t = 1, \ldots, T\right\}.$$

An element c_i of $\mathscr{C}_i$ is written as $\{x_i(t), \Delta v_i(t)\}_1^T$, where both $x_i(t)$ and $\Delta v_i(t)$ are points in R^l. Define.

$$X_i \equiv \{\{x_i(t)\}_1^T \in R^{lT} | \exists \{\Delta v_i(t)\}_1^T;\ \{x_i(t), \Delta v_i(t)\}_1^T \in \mathscr{C}_i\}.$$

We shall assume for any i,

(a.1) $\mathscr{C}_i \subset R^{lT} \times V$,

and there exists $\chi_i \in R^{lT}$ such that $X_i + \{\chi_i\} \subset \Omega^{lT}$,

(a.2) $\mathscr{C}_i$ is convex and closed in R^{2lT}.

Preference Preordering, $\succsim_i$, $i = 1, 2, \ldots, m$:

A binary relation, $\succsim_i$, which will be called a preference preordering, is supposed to be defined on $\mathscr{C}_i$. It is a complete preordering in the sense that,

For any $c \in \mathscr{C}_i$, $c \succsim_i c$ (reflexive)

For any $c, c', c'' \in \mathscr{C}_i$, $[c \succsim_i c'\ \&\ c' \succsim_i c''] \Rightarrow c \succsim_i c''$ (transitive)

For any $c, c' \in \mathscr{C}_i$, either $c \succsim_i c'$ or $c' \succsim_i c$ holds. (complete)

It is usefull to introduce some further concepts: For any $c, c' \in \mathscr{C}_i$, we define

$$c \succ_i c' \Leftrightarrow [c \succsim_i c'\ \&\neg\ c' \succsim_i c],$$

$$c \sim_i c' \Leftrightarrow [c \succsim_i c'\ \&\ c' \succsim_i c].$$

We shall assume for any i, the following conditions hold. $\hat{\mathscr{C}}_i$ in (b.1) is a subset of $\mathscr{C}_i$ and its precise definition will be given in Section VI (p. 54).

(b.1) For any $c_i \in \hat{\mathscr{C}}_i$, there exists $c_i' \in \mathscr{C}_i$ such that $c_i' \succ_i c_i$.

(b.2) For any $c_i' \in \mathscr{C}_i$, both $\{c_i \in \mathscr{C}_i | c_i \succsim_i c_i'\}$ and $\{c_i \in \mathscr{C}_i | c_i' \succsim_i c_i\}$ are closed.

(b.3) For any $c_i' \in \mathscr{C}_i$, $\{c_i \in \mathscr{C}_i | c_i \succsim_i c_i'\}$ is convex.

IV. ECONOMY

Tableau economique:

To sum up, the commodities stream as is illustrated in Fig. I; the vertical arrows indicate inflows to and outflows from the economy, and the horizontal arrows indicate movement or transformation of stocks inside the economy.

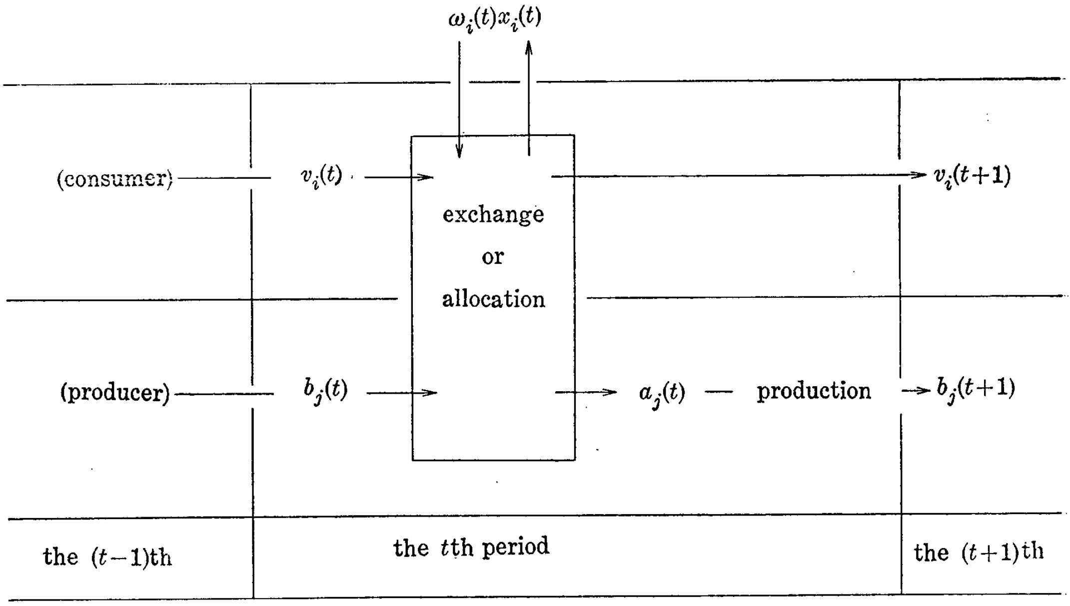

FIGURE I

Quasi-Equilibrium:

The Economy $((\mathscr{C}_i), (\succsim_i), (\{\mathscr{H}_{tj}\}_1^T), (v_i(1)), (\{\omega_i(t)\}_1^T), (b_j(1)), (a_j(T)))$ is said to have a quasi-equilibrium if and only if there exist $\{(a_j^*(t), b_j^*(t+1))\}_1^T \in \mathscr{Y}_j$, $\{(x_i^*(t), \Delta v_i^*(t))\}_1^T \in \mathscr{C}_i$, $\{p^*(t)\}_1^T \in R^{lT}$ such that

(α) for every i, $\{(x_i^*(t), \Delta v_i^*(t))\}_1^T$ is the greatest element of

$$\left\{\{(x_i(t), \Delta v_i(t))\}_1^T \in \mathscr{C}_i \middle| \sum_{t=1}^T p^*(t)(x_i(t) + \Delta v_i(t)) \leq \sum_{t=1}^T p^*(t)\Big(\omega_i(t) + \sum_{j=1}^n \theta_{ij}(b_j^*(t) - a_j^*(t))\Big)\right\} \quad \text{for } \succsim_i \text{ and/or}$$

$$\sum_{t=1}^{T} p^*(t)(x_i^*(t) + \Delta v_i^*(t)) = \sum_{t=1}^{T} p^*(t)\Big(\omega_i(t) + \sum_{j=1}^{n} \theta_{ij}(b_j^*(t) - a_j^*(t))\Big)$$
$$= \mathrm{Min}\left\{\sum_{t=1}^{T} p^*(t)(x_i(t) + \Delta v_i(t)) \middle| \{(x_i(t), \Delta v_i(t))\}_1^T \in \mathscr{C}_i\right\};$$

(β) for every j,

$$\sum_{t=1}^{T} p^*(t)(b_j^*(t) - a_j^*(t))$$
$$= \mathrm{Max}\left\{\sum_{t=1}^{T} p^*(t)(b_j(t) - a_j(t)) \middle| \{(a_j(t), b_j(t+1))\}_1^T \in \mathscr{Y}_j\right\};$$

(γ) for every t,

$$\sum_i (x_i^*(t) + \Delta v_i^*(t) - \omega_i(t)) = \sum_j (b_j^*(t) - a_j^*(t));$$

(δ) $\{p^*(t)\}_1^T \neq 0$.

V. DIGRESSIONS

Define

$$X_i \equiv \{\{x_i(t) + \Delta v_i(t)\}_1^T | \{x_i(t), \Delta v_i(t)\}_1^T \in \mathscr{C}_i\}; \quad X \equiv \sum_i X_i$$
$$Y_j \equiv \{\{b_j(t) - a_j(t)\}_1^T | \{a_j(t), b_j(t+1)\}_1^T \in \mathscr{Y}_j\}; \quad Y \equiv \sum_j Y_j$$
$$\omega_i \equiv \{\omega_i(t)\}_1^T .$$

If preference preordering $\precsim_i$ is defined in X_i for every i, it is well known that there exists a quasi-equilibrium for the economy satisfying the following conditions: (See Debreu [4].)

(a.1) $AX \cap (-AX) = \{0\}$;

For every i

(a.2) X_i is closed and convex,

(b.1) for every consumption x_i in $\hat{X}_i$ there is a consumption in X_i preferred to x_i,

(b.2) for every x_i' in X_i the set $\{x_i \in X_i | x_i \gtrsim_i x_i'\}$ and $\{x_i \in X_i | x_i' \gtrsim_i x_i\}$ are closed in X_i,

(b.3) for every x_i' in X_i, the set $\{x_i \in X_i | x_i \gtrsim_i x_i'\}$ is convex;

(c.1) $(\{\omega\} + Y) \cap X \neq \phi$;

(c.2) there is a closed, convex augmented total production set $\ddot{Y}$ such that, for every i,

$$(\{\omega_i\} + A\ddot{Y} - D) \cap X_i \neq \phi;$$

For every j

(d.1) $0 \in Y_j$;

(d.2) $AX \cap AY = \{0\}$.

Here, by $\hat{X}_i$ we denote the set of all attainable acts for the ith consumer, and by D the smallest cone with vertex 0 owning all points of the form $\sum_i (x_i - \omega_i)$, where $x_i \succ_i \hat{X}_i$ for every i.

The main purpose of Sections II–VI is to explore direct conditions on $\mathscr{C}_i$, $\mathscr{H}_{tj}$, etc, under which there exists a quasi-equilibrium.

VI. EXISTENCE OF A QUASI-EQUILIBRIUM

Since concept V seems unfamiliar, we shall summarize its nature at the beginning of this section. One can easily prove;

Lemma 1

(*i*) *V is a closed convex cone with vertex* 0 *in* R^{lT},

(*ii*) $AV = V$,

(*iii*) *V is not a linear manifold,*

(*iv*) $V \cap (-V) = \{0\}$.

Throughout this paper, we are interested in the economy which satisfies the conditions (a.1)–(a.2), (b.1)–(b.3), (d.1)–(d.5) mentioned in Section III, in addition to the condition

(c) $$[\{\{b(t) - a(t)\}_1^T | \{a(t), b(t+1)\}_1^T \in A\mathscr{Y}\} + \{\omega_i(t)\}_1^T] \cap \{\{x_i(t) + \Delta v_i(t)\}_1^T | \{x_i(t), \Delta v_i(t)\}_1^T \in \mathscr{C}_i\} \neq \phi .$$

A further concept is now introduced:

$$\hat{\mathscr{C}}_i \equiv \Big\{c_i \in \mathscr{C}_i | \exists \{a(t), b(t+1)\} \in \mathscr{Y};\ \exists c_\iota \in \mathscr{C}_\iota \ \text{ for all } \iota \neq i;\ \forall t;\ \sum_i (x_i(t) + \Delta v_i(t) - \omega_i(t)) = b(t) - a(t)\Big\}$$

$$\hat{\mathscr{Y}} \equiv \Big\{y \in \mathscr{Y} | \exists c_i \in \mathscr{C}_i \ \text{ for all } i;\ \forall t;\ \sum_i (x_i(t) + \Delta v_i(t) - \omega_i(t)) = b(t) - a(t)\Big\}.$$

These can be designated as an attainable consumption set for i, and an attainable production set, respectively. With these definitions, the economic interpretation of the condition (b.1) is obvious. We confine out attention to lemmata. The proof of Lemma 4 is almost the same as that of the corresponding lemma in Debreu [3], [4].

Lemma 2: Let C be a compact set in R^l. Then the projecting cone $\Gamma(C)$ of C is closed in R^l.

Proof of Lemma 2 Define the mapping

$$f : R^l\backslash\{0\} \to I \times S\,,$$

where

$$I \equiv (0, \infty)\,,$$
$$S \equiv \{\xi \in R^l \mid ||\xi|| = 1\}\,,$$
$$f(x) = (t, y) \Leftrightarrow x = ty\,.$$

f is clearly a topological mapping. Define also the mapping

$$f' : R^l\backslash\{0\} \to S\,,$$

where

$$f'(x) = y \Leftrightarrow f(x) = (t, y)\,.$$

f' is continuous.

Since $C\backslash\{0\}$ is also compact under the relative topology in $R^l\backslash\{0\}$, $f'(C\backslash\{0\})$ is compact, so that

$$\Gamma(C) = f^{-1}(I \times f'(C\backslash\{0\})) \cup \{0\}$$

is closed in R^l. Q.E.D.

Lemma 3: If the conditions (a.1), (d.1), (d.2), (d.3), (d.5) *hold,* $\hat{\mathscr{C}}_i$ *and* $\hat{\mathscr{Y}}$ *are bounded.*

Proof of Lemma 3 Define a set in $R^{2lT(m+1)}$ as

$$A \equiv ((\mathscr{C}_i), (\mathscr{H}_t)) \cap M$$

where

$i = 1, 2, \ldots, m,$
$t = 1, 2, \ldots, T,$

$$M \equiv \left\{(c_i), (a(t), b(t+1)) \mid \forall\, t\,;\ \sum_i (x_i(t) + \Delta v_i(t) - \omega_i(t)) = b(t) - a(t)\right\}.$$

It is sufficient to prove that $A((\mathscr{C}_i), (\mathscr{H}_t))$ and AM are positively semi-independent. First, we note

$$A((\mathscr{C}_i), (\mathscr{H}_t)) \subset ((A\mathscr{C}_i), (A\mathscr{H}_t))\,,$$

$$AM = A\Big[\Big\{(c_i), (a(t), b(t+1)) \mid \forall\, t\,;\ \sum_i (x_i(t) + \Delta v_i(t)) = b(t) - a(t)\Big\} + \{(\omega_i(t)), 0, 0, 0\}\Big]$$

$$= A\Big\{(c_i), (a(t), b(t+1)) \mid \forall\, t\,;\ \sum_i (x_i(t) + \Delta v_i(t)) = b(t) - a(t)\Big\}\,,$$

so that we need only to show that, if $c_i \in A\mathscr{C}_i$ for $i = 1, \ldots, m$, if $(a(t), b(t+1)) \in A\mathscr{H}_t$ for $t = 1, \ldots, T$, and if $\sum_i (x_i(t) + \Delta v_i(t)) = b(t) - a(t)$ for

$t = 1, \ldots, T$, then $c_i = 0$ for any i and $((a(t), b(t+1)) = 0$ for all t. From (a.1),

$$A\mathscr{C}_i \subset A(X_i \times V) \subset AX_i \times V \subset \Omega^{lT} \times V .$$

From (d.1), (d.2) and (d.5),

$$A\mathscr{H}_t \subset \mathscr{H}_t \subset \Omega^{2l} .$$

(i) $t = 1$,

$$\sum_i (x_i(1) + v_i(1)) \equiv \sum_i (x_i(1) + \Delta v_i(1)) = b(1) - a(1) \equiv -a(1) .$$

Since $a(1), x_i(1), v_i(1) \geqq 0$, we see

$$x_i(1) = v_i(1) = a(1) = 0 .$$

(ii) $t = 2$,

From the result of (i),

$$\Delta v_i(2) = v_i(2) \geqq 0 \text{ by the definition of } V,$$
$$b(2) = 0 \text{ by (d.3)}.$$

We see the condition becomes

$$\sum_i (x_i(2) + v_i(2)) = -a(2) .$$

Again, we obtain

$$x_i(2) = v_i(2) = a(2) = 0 .$$

This continues up to T. Q. E. D.

Lemma 4: If the conditions, (a.2) *and* (d.2), *hold, then* $\hat{\mathscr{C}}_i$ *and* $\hat{\mathscr{Y}}$ *are closed.*

For convenience, we shall denote $p \equiv \{p(t)\}_1^T$. In order to characterize quasi-equilibrium, we introduce a correspondence ψ_i: $R^{lT} \times R \to R^{2lT}$, such that

$$\psi_i(p, w_i) \equiv \begin{cases} \left\{c_i \in \xi_i(p, w) \,|\, \text{The value of } c_i \text{, i.e., } \sum_t p(t)(x_i(t) + \Delta v_i(t)) \text{, is the highest in } \xi_i(p, w_i)\right\} \\ \qquad \text{if } w_i > \text{Max} \left\{\sum_t p(t)(x_i(t) + \Delta v_i(t)) | c_i \in \mathscr{C}_i\right\}, \\ \left\{c_i \in \mathscr{C}_i | \sum_t p(t)(x_i(t) + \Delta v_i(t)) = w_i\right\} \\ \qquad \text{otherwise,} \end{cases}$$

where $\xi_i(p, w_i)$ is the demand correspondence of the ith consumer given price vector p and wealth w_i.

Theorem: The private ownership economy, which satisfies the conditions (a.1)–(a.2), (b.1)–(b.3), (c), (d.1)–(d.5), *has a quasi-equilibrium.*

Proof of Theorem: Outline The theorem can be proved in a way similar to that of Debreu [4], though there are some differences in detail. We shall sketch the proof of the theorem in the case where $\mathscr{C}_i$ is bounded: The case where $\mathscr{C}_i$ is unbounded can be treated easily, as we use Lemma 4.

Let $\mathscr{K}^q(\subset R^{2lT})$ be any hypercube which contains the origin and becomes infinitely large as we let $q \to \infty$. Define

$$\mathscr{Y}_j^q \equiv \mathscr{Y}_j \cap \mathscr{K}^q ,$$

$$\mathscr{Y}^q \equiv \mathscr{Y} \cap (n\mathscr{K}^q) ,$$

$$\pi_j^q(p) \equiv \text{Sup}\left\{\sum_{t=1}^{T} p(t)(b_j(t) - a_j(t))|\{a_j(t), b_j(t+1)\}_1^T \in \mathscr{Y}_j^q\right\} ,$$

$$\pi^q(p) \equiv \text{Max}\left\{\sum_{t=1}^{T} p(t)(b(t) - a(t))|\{a(t), b(t+1)\}_1^T \in \mathscr{Y}^q\right\} ,$$

$$d^q(p) \equiv \pi^q(p) - \sum_{j=1}^{n} \pi_j^q(p) ,$$

$$\eta^q(p) \equiv \left\{\{a(t), b(t+1)\}_1^T \in \mathscr{Y}^q | \sum_{t=1} p(t)(b(t) - a(t)) = \pi^q(p)\right\} ,$$

$$Y \equiv \{\{b(t) - a(t)\}_1^T \in R^{lT}|\{a(t), b(t+1)\}_1^T \in A\mathscr{Y}\} .$$

Several properties of these concepts can now be deduced:

(1) $$d^q(p) \geqq 0 \quad \text{for any } p,$$

since $\sum_j \mathscr{Y}_j^q = \sum_j (\mathscr{Y}_j \cap \mathscr{K}^q) \subset \left(\sum_j \mathscr{Y}_j\right) \cap n\mathscr{K}^q$.

$\eta^q(p)$ is upper semicontinuous at any p.

(2) Y is a convex cone with vertex 0 in R^{lT}, and

(3) Y is not a linear manifold,

since $b(1) - a(1) = -a(1) \leqq 0$ for any $\{b(t) - a(t)\}_1^T \in Y$.

(4) Y is closed in R^{lT} :

To see the closedness of Y, we note that the set

$$Y' \equiv \{\{b(t) - a(t)\}_1^T \in R^{lT}|\{a(t), b(t+1)\}_1^T \in A\mathscr{Y} \cap S\}$$

is compact, where $S \equiv \{y \in R^{2lT}| \, ||y|| = 1\}$, since Y' is an image of a continuous function on the compact set $A\mathscr{Y} \cap S$, so that Y must be closed from Lemma 2.

Next, we consider the polar of Y and define its subset P as,

$$P \equiv \{p \in R^{lT}|pY \leqq 0\} \cap S \equiv \bigcap_{q \in Y} \{p \in R^{lT}|pq \leqq 0\} \cap S ,$$

where

$$S = \{p \in R^{lT} \mid \|p\| = 1\},$$

and a real valued function w_i^q on P as

$$w_i^q(p) \equiv \sum_{t=1}^{T} p(t)\omega_i(t) + \sum_j \theta_{ij}\pi_j^q(p) + \frac{1}{m}d^q(p).$$

Obviously, for any $p \in P$,

$$w_i^q(p) \geqq \sum_t p(t)\omega_i(t),$$

(5) $$\sum_t w_i^q(p) = \sum_i p(t)\omega(t) + \pi^q(p),$$

(6) $$w_i^q(p) \geqq \text{Min}\left\{\sum_{t=1}^{T} p(t)(x_i(t) + \Delta v_i(t)) \mid \{x_i(t), \Delta v_i(t)\}_1^T \in \mathscr{C}_i\right\}.$$

The first inequality follows from (d.5) and (1) above, and the last inequality from (c). Note that Y is a polar of P, because of (2) and (4).

Now let ζ^q: $R^{lT} \to R^{lT}$ be a correspondence such that

$$\zeta^q(p) \equiv \left\{\left\{\sum_i (x_i(t) + \Delta v_i(t)) - \omega(t) - (b(t) - a(t))\right\}_1^T \in R^{lT} \mid c_i \in \psi_i(p, w_i^q(p)), y \in \eta^q(p)\right\}.$$

(7) ζ^q is upper semicontinuous in P,

since both ψ_i and η^q are upper semicontinuous in P, and closedness of a correspondence f: $X \to Y$, together with continuity of a single mapping g: $Y \to Z$, insures closedness of $g \circ f$: $X \to Z$, where Y is compact.

$\zeta^q(p)$ is convex for and $p \in P$.

(8) $p\zeta^q(p) \leqq 0$ for any $p \in P$,

because of (5) and (6).

(9) $\zeta^q(p) \neq \phi$ for all $p \in P$,

because of (6). Since Y is a polar of P, since $\zeta^q(p)$ is always in some compact set because of the compactness of $\mathscr{C}_i$ and $\mathscr{Y}^q$, and since (3), (7), (8), (9) hold, we can apply the theorem in Debreu [2] to our case, to obtain the existence of $p^q \in P$ such that

$$Y \cap \zeta^q(p^q) \neq \phi.$$

If we let z be any point in the intersection,

there exists $y \in A\mathscr{Y}$ such that $z = \{b(t) - a(t)\}_1^T$, and
there exist $c_i^q \in \psi_i(p^q, w_i^q(p^q))$ and $\bar{y}^q \in \eta^q(p^q)$ such that

$$z \equiv \left\{\sum_i (x_i^q(t) + \Delta v_i^q(t)) - \omega(t) - (\bar{b}^q(t) - \bar{a}^q(t))\right\}_1^T.$$

Define

$$y^q \equiv y + \bar{y}^q ,$$

and adopting the similar technique to that in Debreu [4], we see,

$$p^q \in P ,$$
$$y^q \in \mathscr{Y} ,$$
$$c_i^q \in \mathscr{C}_i ,$$
$$\sum_i (x_i^q(t) + \Delta v_i^q(t)) - \omega(t) - (b^q(t) - a^q(t)) = 0 ,$$
$$c_i^q \in \phi_i(p^q, w_i^q(p^q)) ,$$
$$w_i^q(p^q) = p^q\omega_i + \sum_j \theta_{ij}\pi_j^q(p^q) + \frac{1}{m}d^q(p^q) ,$$
$$\sum_{t=1}^T p^q(t)(x_i^q(t) + \Delta v_i^q(t)) = w_i^q(p^q) ,$$
$$\sum_{t=1}^T p^q(t)(b^q(t) - a^q(t)) = \pi^q(p^q) .$$

Finally, the fact that $(p^q, (c_i^q), y^q)$ lies in a compact set $P \times (\hat{\mathscr{C}}_i) \times \hat{\mathscr{Y}}$ for any q enables us to find a convergent subsequence, the limit of which turns out the required quasi-equilibrium point. Q. E. D.

VII. GENERALIZATIONS

So far, we have restricted ourselves to the consideration of a special type of economy in which neither wear and tear of goods nor the factors of uncertainty played an important role. The purpose of this section is to generalize the model so as to take these factors into consideration. We shall first formally present a new model, and then discuss its economic interpretations.

Model:

Our new model differs from the preceding one mainly in two points.

The first point is that, while there were l kinds of commodities for each period in the economy of the preceding model, we now admit that the number of the kinds of commodities at a given period depends upon the period when they exist, so that there are $l(t)$ kinds of commodities at the tth period, where l is now a function of t.

The second point is that we now assume that if a consumer saves $\alpha_i(t)$ ($\in R^{l(t)}$) units of commodities at the tth period, they will turn into $\beta_i(t+1) \equiv F_{ti}(\alpha_i(t))$ ($\in R^{l(t+1)}$) units of commodities at the beginning of the next period, where F_{ti}: $\Omega^{l(t)} \to \Omega^{l(t+1)}$ might be called a transformation function. The concept V can be used no longer, although its essence will still be retained. The new "tableau économique" might be Fig. II. We shall assume for any i and t,

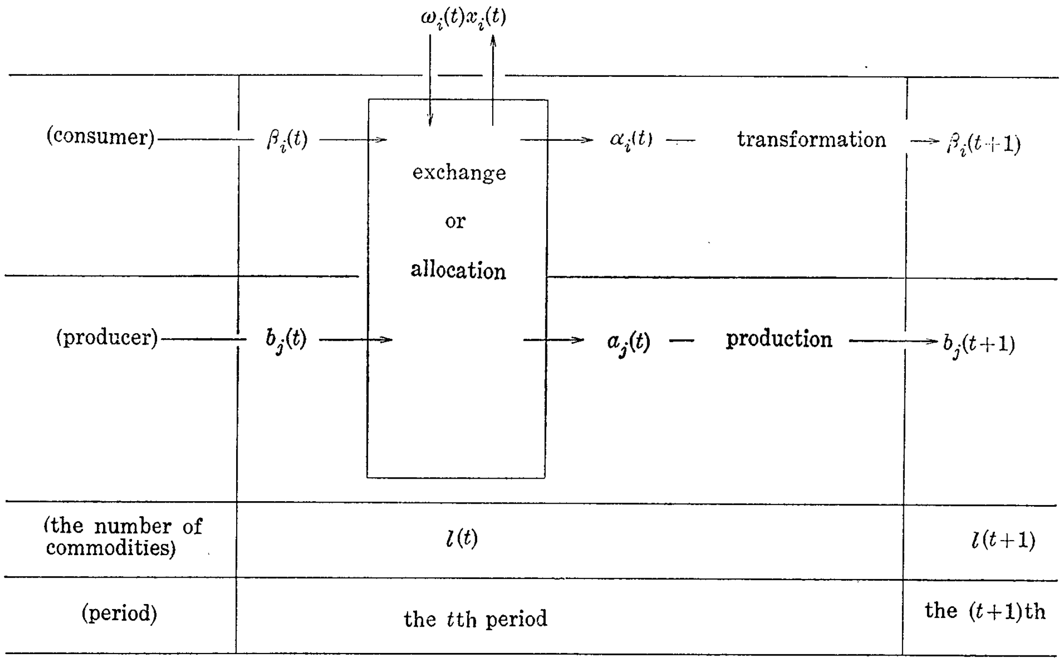

FIGURE II

(*) A single valued function F_{ti}: $\Omega^{l(t)} \longrightarrow \Omega^{l(t+1)}$ is everywhere continuous,

(**) $F_{ti}(0) = 0$.

As a consequence, $\mathscr{C}_i(\in R^{2[l(1)+l(2)+\cdots+l(T)]})$ is a set of all meaningful $\{x_i(t), \alpha_i(t)\}_1^T$, $\succsim_i$ is defined on the new $\mathscr{C}_i$, and the behavior of the ith consumer can be summarized as,

maximizeing his preference,

subject to $\{x_i(t), \alpha_i(t)\}_1^T \in \mathscr{C}_i$,

$$\sum_{t=1}^{T} p(t)(x_i(t) + \alpha_i(t) - F_{t-1,i}(\alpha_i(t-1)))$$
$$\leqq \sum_{t=1}^{T} p(t)\Big(\omega_i(t) + \sum_j \theta_{ij}(b_j(t) - a_j(t))\Big)$$

given $p(t)$, $\omega_i(t)$, $a_j(t)$, $b_j(t) \in R^{l(t)}$, and

$$0 \equiv F_{0i}(\alpha_i(0)) \in R^{l(1)}, \quad 0 \equiv b_j(1), \quad t = 1, \ldots, T, \quad j = 1, \ldots, n.$$

Production sets, producer's behavior and others must also be understood *mutatis mutandis*.

The conditions (*) (**), together with the variants of (a.1)–(d.5), ensure the existence of a quasi-equilibrium. Indeed, one can easily prove it, following the techniques in the proof of Lemmas 3, 4 and Theorem.

Economic Interpretation (I):

As we have noted, commodities were not distinguished according to their ages in the preceding model, so that only a special type of physical depreciation (e.g., diminishing-balance method) could be treated there. We shall remove this restriction with the aid of the transformation function F_{ti}: $\Omega^{l(t)} \to \Omega^{l(t+1)}$ and the production set $\mathscr{H}_{tj} \subset \Omega^{l(t)+l(t+1)}$. The amount $\alpha_{i,h(\tau)}(t)$ of the commodity, "the hth good which is τ years old", will be *transformed* into the amount $\beta_{i,h(\tau+1)}(t+1)$ of *another* commodity, "the hth good which is $(\tau+1)$ years old", (where the shape of the function F_{ti} makes allowences for wear and tear of the good, of course), and the same to capital goods on production side. See von Neumann [9].

Economic Interpretation (II):

We now consider the factors of uncertainty within the framework of our new model. Following Debreu [3] Ch. 7, we define events at any future period, that can be conveniently represented by the verteces of a tree, the number of which is assumed finite.(2) The tth period has supposedly k_t kinds of events, where by an event $e_t \in \{1_t, 2_t, \dots, k_t\}$ atmospheric conditions, natural disasters, technical possibilities...are described for all the preceding periods, in addition to those of the tth period.

A commodity can now be defined by its event (or vertex of the event tree as is drawn in Fig. III, for example), as well as by its physical characteristics, its location and its age.

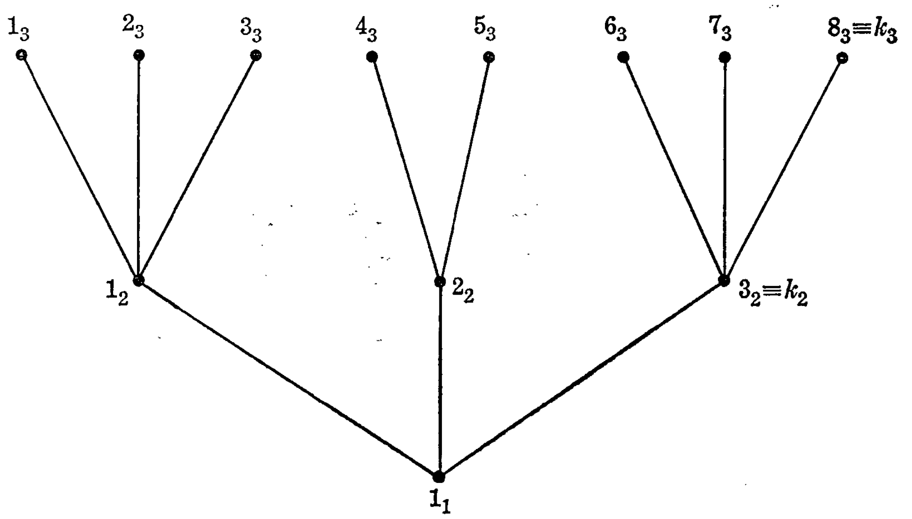

FIGURE III

(2) Radner's definition in his recent paper [11] are not useful in our case. It is in this sence that Debreu's definition of events is convenient when we treat stocks and flows of goods.

Future contracts are made at the beginning of the first period: The *claim* for any future good in the case of realization of an event is determined in the markets. See Debreu [3] 7. 3. in detail.

The transformation function F_{ti} also plays an important role here. Suppose the ith consumer saves in the tth period the $h(\tau, 1_t)$th commodity as much as $\alpha_{ih(\tau,1_t)}$ unit(s), which should be understood as the amount of "the claim for the hth good that is τ years old, in the case of realization of the 1_tth event". Then, it will be *transformed* into the amount $\beta_{i,h(\tau+1,e_{t+1})}(t+1)$ of all the commodities "the claim for the hth good that is $(\tau+1)$ years old, in the case of realization of the e_{t+1}th event", where e_{t+1}'s are compatible with 1_t, that is, e_{t+1} describes the same history up to the tth as 1_t. In the example of Fig. III, if a consumer saves $\alpha_{i,h(\tau,2_2)}(2)$ unit(s) of the $h(\tau, 2_2)$th commodity in the second period, and if there is no depreciation, then he will get in the next period the $h(\tau+1, 4_3)$th commodity and the $h(\tau+1, 5_3)$th, where

$$\beta_{i,h(\tau+1,4_3)}(3) = \beta_{i,h(\tau+1,5_3)}(3) = \alpha_{i,h(\tau,2_2)}(2) .$$

The amount of the commodities $h(\tau+1, 1_3)$, $h(\tau+1, 2_3)$, $h(\tau+1, 3_3)$, $h(\tau+1, 6_3)$, $h(\tau+1, 7_3)$, $h(\tau+1, 8_3)$ might be well determined independent of the amount of the $h(\tau, 2_2)$th.

All the interpretations written above should be applied also to the production side, *mutatis mutandis*.

In closing this section, I should like to retrieve a theorem in my survey article [6]:

Let $V_i = \sum_{s=1}^{S} \pi_{is} U_i$, *where* $(\pi_{i1}, \ldots, \pi_{iS})$ *is a probability vector.* (*Here, we are using the same notations as in Arrow* [1].) *Then,*

(*i*) U_i *is quasi-concave if* V_i *is quasi-concave.*

(*ii*) U_i *is concave if and only if* V_i *is concave.*

It turns out that, if we adopt the Expected-Utility Hypothesis rigorously analyzed first by von Neumann & Morgenstern [10], then the character of *diversifier* in Arrow's sense ensures concavity of his preference function, hence the condition (b.3) in the present paper.

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