慶應義塾大学学術情報リポジトリ
Keio Associated Repository of Academic resouces

| Title | STOCKS AND FLOWS IN STATIC EQUILIBRIUM ANALYSIS |
| :---: | :--- |
| Sub Title |  |
| Author | ICHIISHI，TATSURO |
| Publisher | Keio Economic Society，Keio University |
| Publication year | 1969 |
| Jtitle | Keio economic studies Vol．6，No．2（1969．），p．47－63 |
| JaLC DOI |  |
| Abstract |  |
| Notes |  |
| Genre | Journal Article |
| URL | https：／／koara．lib．keio．ac．jp／xoonips／modules／xoonips／detail．php？koara＿id＝AA00260492－19690002－0 <br> 047 |

慶應義塾大学学術情報リポジトリ（KOARA）に掲載されているコンテンツの著作権は，それぞれの著作者，学会または出版社／発行者に帰属し，その権利は著作権法によって保護されています。引用にあたっては，著作権法を遵守してご利用ください。

The copyrights of content available on the KeiO Associated Repository of Academic resources（KOARA）belong to the respective authors，academic societies，or publishers／issuers，and these rights are protected by the Japanese Copyright Act．When quoting the content，please follow the Japanese copyright act

# STOCKS AND FLOWS IN STATIC EQUILIBRIUM ANALYSIS 

by Tatsuro Ichishi

## I. introduction

If we confine our attention to the existence of a general economic equilibrium and if, moreover, we are satisfied with static analysis, it could be argued that the analysis of general equilibrium systems has virtually been finished. The most complete account of this branch of theory might be found in Debreu's work, see [4]. With this analysis as the essential background, some economists considered factors that had been neglected in the pure models referred to above. Radner [11], for example, following Arrow [1], extended the model in an elegant way in order to give insights into the workings of an economy of an uncertain world. Malinvaud defined intertemporal efficiency in his model [7], [8], and explained the relationship between competitive equilibrium and intertemporal efficient allocation.

However, little attention is paid in these works to the distinction between stocks and flows of commodities. A reasonable interpretation of the models is therefore that no transactions in durable goods may occur; though economic units may buy and sell their services in addition to perishable commodities. But, in the economies that we actually observe, durable goods are transacted, and second-hand markets for them exist, although they are very imperfect and inadequate ones.

The purpose of this paper is to explore how far traditional static methods may be applied to the factors that are essential to an economy which embodies the variable, time. In particular, we shall modify the traditional model in order to introduce into the analysis the sale and purchase of durable goods, as well as of services.

By focusing attention on "stocks and flows", I hope to make the model a stepping-stone to dynamic models that contain money and bonds as well. Indeed it has often been noted that money and bonds, which appear essentially in dynamic economics, must in an uncertain world be regarded as a type of durable goods. (See, e.g., my survey article [6].)

The contents of this paper are divided into two parts: The first part, from Section II to Section VI, deals with a special model, in order to highlight its special features and characteristics. The nature of a stock, that its amount is nonnegative and that one cannot obtain an infinite amount of goods by saving finite amounts previously, will be shown to be part of the sufficient conditions which ensure the existence of a quasi-equilibrium. In the second
part, Section VII, a general case will be discussed, in which the factors of uncertainty and depreciation are fully considered. It will be shown that, when analyzing stocks and flows of goods, Debreu's definition of events in [3] Ch. 7 is more useful than that in Radner [11].

The following mathematical notations will be employed throughout the paper:
$R^{n} \quad: n$-dimensional Euclidian space.
$\Omega^{n} \quad$ : nonnegative orthant of $R^{n}$.
$\left\|\|:\right.$ norm in $R^{n}$.
0 : zero element.
For any set $X$,
$A X \quad$ : asymptotic cone of $X$ with vertex 0 .
$\{\hat{\xi}(t)\}_{1}^{T}: \quad(\xi(1), \xi(2), \ldots, \xi(T))$.

## II. basic concepts (I)

Commodities, $h=1,2, \ldots, l$ :
We shall suppose the economy to have $l$ kinds of commodities, where commodities are to be interpreted broadly, including perishable as well as durable goods and services, distinguished possibly according to location.

## Future Market and Prices:

The economy is supposed to have $T$ periods. Spot contracts for the first period, and future contracts for the subsequent periods up to the $T$ th are made at the beginning of the first period. A vector $p(1)$ of spot prices and vectors. $p(2), \ldots, p(T)$ of prices for future contracts are determined in such a way that total demands and total supplies balance, where, by the $h$ th coordinate $p_{h}(t)$ of $p(t) \in R^{l}$, we denote the future (or spot, if $t=1$ ) price of the $h$ th commodity of the $t$ th period. It is also assumed that pure competition prevails. in all the markets.

Producers, $j=1,2, \ldots, n$ :
It is assumed that there are $n$ agents called producers in the economy: Each producer determines, at the beginning of the first period, the production plan for the $t$ th period when the $j$ th producer uses commodities $a_{j}(t) \in R^{l}$ to produce commodities $b_{j}(t+1) \in R^{l}$ at the next period $(t=1,2, \ldots, T)$. Here, each $h$ th coordinate of the vectors $a_{j}(t), b_{j}(t+1)$ denotes the stock amount of the $h$ th commodity.

The relation between the input vector $a_{j}(t)$ and the output vector $b_{j}(t+1)$. depends upon the level of technology, the feasible set of which will be written as $\mathscr{K}_{t j}$, i.e., $\left(a_{j}(t), b_{j}(t+1)\right) \in \mathscr{\mathscr { E }}_{t j}$ if and only if such a production process.
is technologically feasible. Technological progress of the disembodied type can be explained by the relation

$$
\mathscr{\mathscr { L }}_{1 j} \subset \mathscr{\mathscr { L }}_{2 j} \subset \cdots \subset \mathscr{\mathscr { H }}_{T j}
$$

although we do not necessarily assume this throughout this paper.
The content of inputs and outputs in this context should be understood in the same way as in von Neumann [9]. Outputs include intermediate products and remaining capital goods in addition to production goods. The diminishingbalance method of depreciation might well be explained by the shape of $\mathscr{\mathscr { H }}_{t j}$ combined with the assumption of constant returns to scale, although we do not necessarily assume this in any context: If the $h$ th commodity is a capital good, then

$$
a_{j h}(t)-b_{j h}(t+1)
$$

is the amount of its physical depreciation. ${ }^{(1)}$
Each producer is supposed to behave in such a way as to maximize his profit over $T$ periods, subject to technological constraints, given spot prices and future prices. More specifically, the $j$ th producer determines $\left\{a_{j}(t), b_{j}(t+1)\right\}_{1}^{T}$,
to maximize $\sum_{t=1}^{T} p(t)\left(b_{j}(t)-a_{j}(t)\right)$,
subject to $\left(a_{j}(t), b_{j}(t+1)\right) \in \mathscr{\mathscr { E }}_{t j}, \quad t=1, \ldots, T$,
given $b_{j}(1)=0$ and $\{p(t)\}_{1}^{T}$.
Consumers, $i=1,2, \ldots, m$ :
It is assumed that there are $m$ agents called consumers in the economy: Each consumer determines, at the beginning of the first period, his consumption-saving-expenditure plan for the $t$ th period when the $i$ th consumer demands commodities as much as $x_{i}(t)+v_{i}(t+1) \in R^{l}$, so that he can then consume $x_{i}(t)$ units of commodities and save $v_{i}(t+1)$ units for the next period $(t=1$, $\ldots, T)$.
The meaning of the amount $x_{i}(t)$ of consumption flow, and the relation between $x_{i}(t)$ and $v_{i}(t+1)$ should be noted: If the $h$ th commodity is a durable good, $x_{i h}(t)$ means the amount of physical depreciation due to the fact that the $i$ th consumer uses the good at the $t$ th period as much as $x_{i h}(t)+v_{i n}(t+1)$, while, if it is a perishable good or service, $v_{i n}(t+1)=0$. We thus have completely durable goods (i.e., $x_{i h}(t)=0$ ) on the one hand, completely perishable goods or services (i.e., $v_{i h}(t+1)=0$ ) on the other hand, and commodities in various degrees between the two poles, the degrees being explained below

[^0]by the shape of the consumption set $\mathscr{C}_{i}$.
Define, for convenience,
$$
\Delta v_{i}(t) \equiv v_{i}(t+1)-v_{i}(t)
$$
and his problem is to choose
$$
c_{i} \equiv\left\{x_{i}(t), \Delta v_{i}(t)\right\}_{1}^{T} \in R^{22 T}
$$

By the subset $\mathscr{C}_{i}$ of $R^{2 l T}$, we denote the set of all such choices which are consistent with the physical nature of commodities and that of the $i$ th consumer. The general condition $v_{i}(t) \geqq 0$, for example, must be imposed on the set $\mathscr{C}_{i}$, so that if $c_{i} \in \mathscr{C}_{i}$, then $\sum_{\tau=1}^{t} \Delta v_{i}(\tau)+v_{i}(1) \geqq 0$ for $t=1, \ldots, T$. But some components of $x_{i}(t)$ may be negative, since the amount of services supplied by a consumer is measured in terms of a negative amount, as usual.

These considerations about $\mathscr{C}_{i}$ will rationalize the condition (a.1) shown in the next section.

Each consumer is assumed to have preference preordering on $\mathscr{C}_{i}$, which we denote by $\gtrsim_{i} . \quad c_{i} \gtrsim_{i} c_{i}^{\prime}$ means that the $i$ th consumer prefers $c_{i}$ to $c_{i}^{\prime}$, or is indifferent between the two.

Now, let $\omega_{i}(t) \in R^{l}$ be a commodity vector that "Nature" gives to the $i$ th consumer at the $t$ th period, and let $\theta_{i j}$ be the ratio of the profit of the $j$ th producer which is a part of the wealth of the $i$ th consumer. Each consumer is supposed to behave in such a way as to maximize his utility subject to his wealth constraint, given spot prices and future prices. More specifically, the $i$ th cosumer determines $c_{i} \equiv\left\{x_{i}(t), \Delta v_{i}(t)\right\}_{1}^{T}$,
to maximize his preference,
subject to $c_{i} \in \mathscr{C}_{i}$,

$$
\sum_{t=1}^{T} p(t)\left(x_{i}(t)+\Delta v_{i}(t)\right) \leqq \sum_{t=1}^{T} p(t)\left(\omega_{i}(t)+\sum_{j=1}^{n} \theta_{i j}\left(b_{j}(t)-a_{j}(t)\right)\right)
$$

given $p(t), \omega_{i}(t), a_{j}(t), b_{j}(t), t=1, \ldots, T, j=1, \ldots, n$, and $v_{i}(1)$.
Finally, when we consider the role of $\omega_{i}(1)$, we see that we can let, without loss of generality,

$$
v_{i}(1)=0 \quad \text { for any } i
$$

Also, since we are considering a private ownership economy, we shall assume,

$$
b_{j}(1)=0 \quad \text { for } \quad j=1, \ldots, n
$$

## III. basic concepts (II)

In this section we will formulate some purely mathematical concepts, the economic interpretation of which is straightforward to any reader who has read the preceding section.

Production Set, $\mathscr{Z}_{t j}, \quad t=1,2, \ldots, T, \quad j=1,2, \ldots, n$ :
We are interested in a subset $\mathscr{\mathscr { C }}_{t j}$ of $R^{2 l}$, the element of which is written as $\left(a_{j}(t), b_{j}(t+1)\right)$, where both $a_{j}(t)$ and $b_{j}(t+1)$ are points in $R^{l}$. Define:

$$
\begin{gathered}
\mathscr{Y}_{j} \equiv\left\{\left\{a_{j}(t), b_{j}(t+1)\right\}_{1}^{T} \in R^{2 l T} \mid\left(a_{j}(t), b_{j}(t+1)\right) \in \mathscr{H}_{t j}, \quad t=1, \ldots, T\right\}, \\
\mathscr{H}_{t} \equiv \sum_{j=1}^{n} \mathscr{\mathscr { H }}_{t j}, \quad \mathscr{Y} \equiv \sum_{j=1}^{n} \mathscr{Y}_{j} .
\end{gathered}
$$

We shall assume for any $t$ and $j$,
(d.1) $\mathscr{H}_{t} \subset \Omega^{2 l}$,
(d.2) $\mathscr{H}_{t}$ is convex and closed in $R^{2 l}$,
(d.3) $(0, b(t+1)) \in \mathscr{H}_{t} \Rightarrow b(t+1)=0$,
(d.4) $A \mathscr{Y}$ does not degenerate to 0 ,
(d.5) $\mathscr{H}_{t j} \ni 0$.

Consumption Set, $\mathscr{C}_{i}, \quad i=1,2, \ldots, m$ :
Let $\mathscr{C}_{i}$ be a subset of $R^{l T} \times V$, where, if we define $\Delta v(t) \equiv v(t+1)-v(t)$ for any $t$,

$$
V \equiv\left\{\{\Delta v(t)\}_{1}^{T} \in R^{l T} \mid \sum_{\tau=1}^{t} \Delta v(\tau) \geqq 0, \quad \text { for any } t=1, \ldots, T\right\}
$$

An element $c_{i}$ of $\mathscr{C}_{i}$ is written as $\left\{x_{i}(t), \Delta v_{i}(t)\right\}_{1}^{T}$, where both $x_{i}(t)$ and $\Delta v_{i}(t)$ are points in $R^{l}$. Define.

$$
X_{i} \equiv\left\{\left\{x_{i}(t)\right\}_{1}^{T} \in R^{l T} \mid \exists\left\{\Delta v_{i}(t)\right\}_{1}^{T} ;\left\{x_{i}(t), \Delta v_{i}(t)\right\}_{1}^{T} \in \mathscr{C}_{i}\right\}
$$

We shall assume for any $i$,
(a.1) $\mathscr{C}_{i} \subset R^{l T} \times V$,
and there exists $\chi_{i} \in R^{l T}$ such that $X_{i}+\left\{\chi_{i}\right\} \subset \Omega^{l T}$,
(a.2) $\mathscr{C}_{i}$ is convex and closed in $R^{2 l T}$.

Preference Preordering, $\gtrsim_{i}, \quad i=1,2, \ldots, m$ :
A binary relation, $\gtrsim_{i}$, which will be called a preference preordering, is supposed to be defined on $\mathscr{C}_{i}$. It is a complete preordering in the sense that,

For any $c \in \mathscr{C}_{i}, c \gtrsim_{i} c \quad$ (reflexive)
For any $c, c^{\prime}, c^{\prime \prime} \in \mathscr{C}_{i},\left[c \gtrsim_{i} c^{\prime} \& c^{\prime} \gtrsim_{i} c^{\prime \prime}\right] \Rightarrow c \gtrsim_{i} c^{\prime \prime} \quad$ (transitive)
For any $c, c^{\prime} \in \mathscr{C}_{i}$, either $c \gtrsim_{i} c^{\prime}$ or $c^{\prime} \gtrsim_{i} c$ holds. (complete)
It is usefull to introduce some further concepts: For any $c, c^{\prime} \in \mathscr{C}_{i}$, we define

$$
\begin{aligned}
& c \succ_{i} c^{\prime} \Leftrightarrow\left[\begin{array}{lll}
c \gtrsim_{i} c^{\prime} \& \neg c^{\prime} \gtrsim_{i} c
\end{array},\right. \\
& c \sim_{i} c^{\prime} \Leftrightarrow\left[\begin{array}{lll}
c \gtrsim_{i} c^{\prime} & \& & c^{\prime} \gtrsim_{i} c
\end{array}\right] .
\end{aligned}
$$

We shall assume for any $i$, the following conditions hold. $\hat{\mathscr{C}}_{i}$ in (b.1) is a subset of $\mathscr{C}_{i}$ and its precise definition will be given in Section VI (p. 54).
(b.1) For any $c_{i} \in \hat{\mathscr{C}}_{i}$, there exists $c_{i}^{\prime} \in \mathscr{C}_{i}$ such that $c_{i}^{\prime}>_{i} c_{i}$.
(b.2) For any $c_{i}^{\prime} \in \mathscr{C}_{i}$, both $\left\{c_{i} \in \mathscr{C}_{i} \mid c_{i} \gtrsim_{i} c_{i}^{\prime}\right\}$ and $\left\{c_{i} \in \mathscr{C}_{i} \mid c_{i}^{\prime} \gtrsim_{i} c_{i}\right\}$ are closed.
(b.3) For any $c_{i}^{\prime} \in \mathscr{C}_{i},\left\{c_{i} \in \mathscr{C}_{i} \mid c_{i} \gtrsim_{i} c_{i}^{\prime}\right\}$ is convex.

## IV. ECONOMY

## Tableau economique:

To sum up, the commodities stream as is illustrated in Fig. I; the vertical arrows indicate inflows to and outflows from the economy, and the horizontal arrows indicate movement or transformation of stocks inside the economy.


FIGURE I

## Quasi-Equilibrium:

The Economy $\left(\left(\mathscr{C}_{i}\right),\left(\gtrsim_{i}\right),\left(\left\{\mathscr{\mathscr { C }}_{t j}\right\}_{1}^{T}\right),\left(v_{i}(1)\right),\left(\left\{\omega_{i}(t)\right\}_{1}^{T}\right),\left(b_{j}(1)\right),\left(a_{j}(T)\right)\right)$ is said to have a quasi-equilibrium if and only if there exist $\left\{\left(a_{j}^{*}(t), b_{j}^{*}(t+1)\right)\right\}_{1}^{T} \in \mathscr{Y}_{j}$, $\left\{\left(x_{i}^{*}(t), \Delta v_{i}^{*}(t)\right)\right\}_{1}^{T} \in \mathscr{C}_{i},\left\{p^{*}(t)\right\}_{1}^{T} \in R^{l T}$ such that
( $\alpha$ ) for every $i,\left\{\left(x_{i}^{*}(t), \Delta v_{i}^{*}(t)\right)\right\}_{1}^{T}$ is the greatest element of

$$
\begin{aligned}
& \left\{\left\{\left(x_{i}(t), \Delta v_{i}(t)\right)\right\}_{1}^{T} \in \mathscr{C}_{i} \mid \sum_{t=1}^{T} p^{*}(t)\left(x_{i}(t)+\Delta v_{i}(t)\right)\right. \\
& \left.\quad \leqq \sum_{t=1}^{T} p^{*}(t)\left(\omega_{i}(t)+\sum_{j=1}^{n} \theta_{i j}\left(b_{j}^{*}(t)-a_{j}^{*}(t)\right)\right)\right\} \quad \text { for } \gtrsim_{i} \text { and/or }
\end{aligned}
$$

$$
\begin{gathered}
\sum_{t=1}^{T} p^{*}(t)\left(x_{i}^{*}(t)+\Delta v_{i}^{*}(t)\right)=\sum_{t=1}^{T} p^{*}(t)\left(\omega_{i}(t)+\sum_{j=1}^{n} \theta_{i j}\left(b_{j}^{*}(t)-a_{j}^{*}(t)\right)\right) \\
=\operatorname{Min}\left\{\sum_{t=1}^{T} p^{*}(t)\left(x_{i}(t)+\Delta v_{i}(t)\right) \mid\left\{\left(x_{i}(t), \Delta v_{i}(t)\right)\right\}_{1}^{T} \in \mathscr{C}_{i}\right\}
\end{gathered}
$$

( $\beta$ ) for every $j$,

$$
\begin{aligned}
& \sum_{t=1}^{T} p^{*}(t)\left(b_{j}^{*}(t)-a_{j}^{*}(t)\right) \\
& \quad=\operatorname{Max}\left\{\sum_{t=1}^{T} p^{*}(t)\left(b_{j}(t)-a_{j}(t)\right) \mid\left\{\left(a_{j}(t), b_{j}(t+1)\right)\right\}_{1}^{T} \in \mathscr{Y}_{j}\right\}
\end{aligned}
$$

( $\gamma$ ) for every $t$,

$$
\sum_{i}\left(x_{i}^{*}(t)+\Delta v_{i}^{*}(t)-\omega_{i}(t)\right)=\sum_{j}\left(b_{j}^{*}(t)-a_{j}^{*}(t)\right)
$$

( $\delta) \quad\left\{p^{*}(t)\right\}_{1}^{T} \neq 0$.

## V. digressions

Define

$$
\begin{aligned}
X_{i} & \equiv\left\{\left\{x_{i}(t)+\Delta v_{i}(t)\right\}_{1}^{T} \mid\left\{x_{i}(t), \Delta v_{i}(t)\right\}_{1}^{T} \in \mathscr{C}_{i}\right\} ; \quad X \equiv \sum_{i} X_{i} \\
Y_{j} & \equiv\left\{\left\{b_{j}(t)-a_{j}(t)\right\}_{1}^{T} \mid\left\{a_{j}(t), b_{j}(t+1)\right\}_{1}^{T} \in \mathscr{Y}_{j}\right\} ; \quad Y \equiv \sum_{j} Y_{j} \\
\omega_{i} & \equiv\left\{\omega_{i}(t)\right\}_{1}^{T} .
\end{aligned}
$$

If preference preordering $\gtrsim_{i}$ is defined in $X_{i}$ for every $i$, it is well known that there exists a quasi-equilibrium for the economy satisfying the following conditions: (See Debreu [4].)
(a.1) $A X \cap(-A X)=\{0\} ;$

For every $i$
(a.2) $X_{i}$ is closed and convex,
(b.1) for every consumption $x_{i}$ in $\hat{X}_{i}$ there is a consumption in $X_{i}$ preferred to $x_{i}$,
(b.2) for every $x_{i}^{\prime}$ in $X_{i}$ the set $\left\{x_{i} \in X_{i} \mid x_{i} \gtrsim_{i} x_{i}^{\prime}\right\}$ and $\left\{x_{i} \in X_{i} \mid x_{i}^{\prime} \gtrsim_{i} x_{i}\right\}$ are closed in $X_{i}$,
(b.3) for every $x_{i}^{\prime}$ in $X_{i}$, the set $\left\{x_{i} \in X_{i} \mid x_{i} \gtrsim_{i} x_{i}^{\prime}\right\}$ is convex;
(c.1) $(\{\omega\}+Y) \cap X \neq \phi$;
(c.2) there is a closed, convex augmented total production set $\ddot{Y}$ such that, for every $i$,

$$
\left(\left\{\omega_{i}\right\}+A \ddot{Y}-D\right) \cap X_{i} \neq \phi ;
$$

For every $j$
(d.1) $0 \in Y_{j}$;
(d.2) $A X \cap A Y=\{0\}$.

Here, by $\hat{X}_{i}$ we denote the set of all attainable acts for the $i$ th consumer, and by $D$ the smallest cone with vertex 0 owning all points of the form $\sum_{i}\left(x_{i}-\omega_{i}\right)$, where $x_{i} \succ_{i} \hat{X}_{i}$ for every $i$.

The main purpose of Sections II-VI is to explore direct conditions on $\mathscr{C}_{i}$, $\mathscr{E}_{t j}$, etc, under which there exists a quasi-equilibrium.

## VI. EXISTENCE OF A QUASI-EQUILIBRIUM

Since concept $V$ seems unfamiliar, we shall summarize its nature at the beginning of this section. One can easily prove;

## Lemma 1

(i) $V$ is a closed convex cone with vertex 0 in $R^{l T}$,
(ii) $A V=V$,
(iii) $V$ is not a linear manifold,
(iv) $V \cap(-V)=\{0\}$.

Throughout this paper, we are interested in the economy which satisfies the conditions (a.1)-(a.2), (b.1)-(b.3), (d.1)-(d.5) mentioned in Section III, in addition to the condition

$$
\begin{align*}
& {\left[\left\{\{b(t)-a(t)\}_{1}^{T} \mid\{a(t), b(t+1)\}_{1}^{T} \in A \mathscr{Y}\right\}+\left\{\omega_{i}(t)\right\}_{1}^{T}\right]}  \tag{c}\\
& \\
& \quad \cap\left\{\left\{x_{i}(t)+\Delta v_{i}(t)\right\}_{1}^{T} \mid\left\{x_{i}(t), \Delta v_{i}(t)\right\}_{1}^{T} \in \mathscr{C}_{i}\right\} \neq \phi .
\end{align*}
$$

A further concept is now introduced:

$$
\begin{aligned}
\hat{\mathscr{C}}_{i} \equiv & \left\{c_{i} \in \mathscr{C}_{i} \mid \exists\{a(t), b(t+1)\} \in \mathscr{Y} ; \exists c_{\imath} \in \mathscr{C}, \text { for all } \iota \neq i\right. \\
& \left.\forall t ; \sum_{i}\left(x_{i}(t)+\Delta v_{i}(t)-\omega_{i}(t)\right)=b(t)-a(t)\right\} \\
\hat{\hat{Y}} \equiv & \left\{y \in \mathscr{Y} \mid \exists c_{i} \in \mathscr{C}_{i} \text { for all } i\right. \\
& \left.\forall t ; \sum_{i}\left(x_{i}(t)+\Delta v_{i}(t)-\omega_{i}(t)\right)=b(t)-a(t)\right\}
\end{aligned}
$$

These can be designated as an attainable consumption set for $i$, and an attainable production set, respectively. With these definitions, the economic interpretation of the condition (b.1) is obvious. We confine out attention to lemmata. The proof of Lemma 4 is almost the same as that of the corresponding lemma in Debreu [3], [4].

Lemma 2: Let $C$ be a compact set in $R^{l}$. Then the projecting cone $\Gamma(C)$ of $C$ is closed in $R^{l}$.

Proof of Lemma 2 Define the mapping

$$
f: R^{\imath} \backslash\{0\} \rightarrow I \times S,
$$

where

$$
\begin{aligned}
I & \equiv(0, \infty), \\
S & \equiv\left\{\xi \in R^{l}\|\xi\|=1\right\}, \\
f(x) & =(t, y) \Leftrightarrow x=t y
\end{aligned}
$$

$f$ is clearly a topological mapping. Define also the mapping

$$
f^{\prime}: R^{l} \backslash\{0\} \rightarrow S,
$$

where

$$
f^{\prime}(x)=y \Leftrightarrow f(x)=(t, y) .
$$

$f^{\prime}$ is continuous.
Since $C \backslash\{0\}$ is also compact under the relative topology in $R^{l} \backslash\{0\}, f^{\prime}(C \backslash\{0\})$ is compact, so that

$$
\Gamma(C)=f^{-1}\left(I \times f^{\prime}(C \backslash\{0\})\right) \cup\{0\}
$$

is closed in $R^{l}$.
Q.E.D.

Lemma 3: If the conditions (a.1), (d.1), (d.2), (d.3), (d.5) hold, $\hat{\mathscr{C}}_{i}$ and $\hat{\mathscr{Y}}$ are bounded.

Proof of Lemma 3 Define a set in $R^{2 l T(m+1)}$ as

$$
A \equiv\left(\left(\mathscr{C}_{i}\right),\left(\mathscr{\mathscr { O }}_{t}\right)\right) \cap M
$$

where

$$
\begin{aligned}
i & =1,2, \ldots, m \\
t & =1,2, \ldots, T \\
M & \equiv\left\{\left(c_{i}\right),(a(t), b(t+1)) \mid \forall t ; \sum_{i}\left(x_{i}(t)+\Delta v_{i}(t)-\omega_{i}(t)\right)=b(t)-a(t)\right\}
\end{aligned}
$$

It is sufficient to prove that $A\left(\left(\mathscr{C}_{i}\right),\left(\mathscr{E}_{t}\right)\right)$ and $A M$ are positively semiindependent. First, we note

$$
\begin{aligned}
& A\left(\left(\mathscr{C}_{i}\right),\left(\mathscr{C}_{t}\right)\right) \subset\left(\left(A \mathscr{C}_{i}\right),\left(A \mathscr{\mathscr { O }}_{t}\right)\right), \\
& A M= A\left[\left\{\left(c_{i}\right),(a(t), b(t+1)) \mid \forall t ; \sum_{i}\left(x_{i}(t)+\Delta v_{i}(t)\right)=b(t)-a(t)\right\}\right. \\
&\left.+\left\{\left(\omega_{i}(t)\right), 0,0,0\right\}\right] \\
&= A\left\{\left(c_{i}\right),(a(t), b(t+1)) \mid \forall t ; \sum_{i}\left(x_{i}(t)+\Delta v_{i}(t)\right)=b(t)-a(t)\right\},
\end{aligned}
$$

so that we need only to show that, if $c_{i} \in A \mathscr{C}_{i}$ for $i=1, \ldots, m$, if $(a(t)$, $b(t+1)) \in A \mathscr{H}_{t}$ for $t=1, \ldots, T$, and if $\sum_{i}\left(x_{i}(t)+\Delta v_{i}(t)\right)=b(t)-a(t)$ for
$t=1, \ldots, T$, then $c_{i}=0$ for any $i$ and $((a(t), b(t+1))=0$ for all $t$. From (a.1),

$$
A \mathscr{C}_{i} \subset A\left(X_{i} \times V\right) \subset A X_{i} \times V \subset \Omega^{l T} \times V
$$

From (d.1), (d.2) and (d.5),

$$
A \mathscr{H}_{t} \subset \mathscr{K}_{t} \subset \Omega^{2 l}
$$

(i) $t=1$,

$$
\sum_{i}\left(x_{i}(1)+v_{i}(1)\right) \equiv \sum_{i}\left(x_{i}(1)+\Delta v_{i}(1)\right)=b(1)-a(1) \equiv-a(1) .
$$

Since $a(1), x_{i}(1), v_{i}(1) \geqq 0$, we see

$$
x_{i}(1)=v_{i}(1)=a(1)=0
$$

(ii) $t=2$,

From the result of (i),

$$
\begin{aligned}
& \Delta v_{i}(2)=v_{i}(2) \geqq 0 \text { by the definition of } V, \\
& b(2)=0 \text { by }(\text { d. } 3) .
\end{aligned}
$$

We see the condition becomes

$$
\sum_{i}\left(x_{i}(2)+v_{i}(2)\right)=-a(2) .
$$

Again, we obtain

$$
x_{i}(2)=v_{i}(2)=a(2)=0
$$

This continues up to $T$. Q.E.D.

Lemma 4: If the conditions, (a.2) and (d.2), hold, then $\hat{\mathscr{C}}_{i}$ and $\hat{\mathscr{Y}}$ are closed.
For convenience, we shall denote $p \equiv\{p(t)\}_{1}^{T}$. In order to characterize quasi-equilibrium, we introduce a correspondence $\psi_{i}: \quad R^{l T} \times R \rightarrow R^{2 l T}$, such that

$$
\psi_{i}\left(p, w_{i}\right) \equiv\left\{\begin{array}{r}
\left\{c _ { i } \in \xi _ { i } ( p , w ) | \text { The value of } c _ { i } , \text { i.e., } \sum _ { t } p ( t ) \left(x_{i}(t)\right.\right. \\
\left.\left.\quad+\Delta v_{i}(t)\right), \text { is the highest in } \xi_{i}\left(p, w_{i}\right)\right\} \\
\text { if } w_{i}>\operatorname{Max}\left\{\sum_{t} p(t)\left(x_{i}(t)+\Delta v_{i}(t)\right) \mid c_{i} \in \mathscr{C}_{i}\right\} \\
\left\{c_{i} \in \mathscr{C}_{i} \mid \sum_{t} p(t)\left(x_{i}(t)+\Delta v_{i}(t)\right)=w_{i}\right\} \\
\text { otherwise }
\end{array}\right.
$$

where $\xi_{i}\left(p, w_{i}\right)$ is the demand correspondence of the $i$ th consumer given price vector $p$ and wealth $w_{i}$.

Theorem: The private ownership economy, which satisfies the conditions (a.1)(a.2), (b.1)-(b.3), (c), (d.1)-(d.5), has a quasi-equilibrium.

Proof of Theorem: Outline The theorem can be proved in a way similar to that of Debreu [4], though there are some differences in detail. We shall sketch the proof of the theorem in the case where $\mathscr{C}_{i}$ is bounded: The case where $\mathscr{C}_{i}$ is unbounded can be treated easily, as we use Lemma 4.

Let $\mathscr{K}^{q}\left(\subset R^{2 l T}\right)$ be any hypercube which contains the origin and becomes infinitely large as we let $q \rightarrow \infty$. Define

$$
\begin{aligned}
\mathscr{Y}_{j}^{q} & \equiv \mathscr{Y}_{j} \cap \mathscr{K}^{q}, \\
\mathscr{Y}^{q} & \equiv \mathscr{Y}^{q} \cap\left(n \mathscr{K}^{q}\right), \\
\pi_{j}^{q}(p) & \equiv \operatorname{Sup}\left\{\sum_{t=1}^{T} p(t)\left(b_{j}(t)-a_{j}(t)\right) \mid\left\{a_{j}(t), b_{j}(t+1)\right\}_{1}^{T} \in \mathscr{Y}_{j}^{q}\right\}, \\
\pi^{q}(p) & \equiv \operatorname{Max}\left\{\sum_{t=1}^{T} p(t)(b(t)-a(t)) \mid\{a(t), b(t+1)\}_{1}^{T} \in \mathscr{Y}^{q}\right\}, \\
d^{q}(p) & \equiv \pi^{q}(p)-\sum_{j=1}^{n} \pi_{j}^{q}(p), \\
\eta^{q}(p) & \equiv\left\{\{a(t), b(t+1)\}_{1}^{T} \in \mathscr{Y}^{q} \mid \sum_{t=1} p(t)(b(t)-a(t))=\pi^{q}(p)\right\}, \\
Y & \equiv\left\{\{b(t)-a(t)\}_{1}^{T} \in R^{l T} \mid\{a(t), b(t+1)\}_{1}^{T} \in A \mathscr{Y}\right\} .
\end{aligned}
$$

Several properties of these concepts can now be deduced:

$$
\begin{equation*}
d^{q}(p) \geqq 0 \quad \text { for any } p \tag{1}
\end{equation*}
$$

since $\sum_{j} \mathscr{Y}_{j}^{q}=\sum_{j}\left(\mathscr{Y}_{j} \cap \mathscr{K}^{q}\right) \subset\left(\sum_{j} \mathscr{Y}_{j}\right) \cap n \mathscr{K}^{q}$.
$\eta^{q}(p)$ is upper semicontinuous at any $p$.
$Y$ is a convex cone with vertex 0 in $R^{l r}$, and $Y$ is not a linear manifold,
since $b(1)-a(1)=-a(1) \leqq 0$ for any $\{b(t)-a(t)\}_{1}^{r} \in Y$.

$$
\begin{equation*}
Y \text { is closed in } R^{l T}: \tag{4}
\end{equation*}
$$

To see the closedness of $Y$, we note that the set

$$
Y^{\prime} \equiv\left\{\{b(t)-a(t)\}_{1}^{T} \in R^{l T} \mid\{a(t), b(t+1)\}_{1}^{T} \in A \mathscr{Y} \cap S\right\}
$$

is compact, where $S \equiv\left\{y \in R^{2 l T} \mid\|y\|=1\right\}$, since $Y^{\prime}$ is an image of a continuous function on the compact set $A \mathscr{Y} \cap S$, so that $Y$ must be closed from Lemma 2.

Next, we consider the polar of $Y$ and define its subset $P$ as,

$$
P \equiv\left\{p \in R^{l T} \mid p Y \leqq 0\right\} \cap S \equiv \bigcap_{q \in Y}\left\{p \in R^{l \tau} \mid p q \leqq 0\right\} \cap S
$$

where

$$
S=\left\{p \in R^{l T} \mid\|p\|=1\right\}
$$

and a real valued function $w_{i}^{q}$ on $P$ as

$$
w_{i}^{q}(p) \equiv \sum_{t=1}^{T} p(t) \omega_{i}(t)+\sum_{j} \theta_{i j} \pi_{j,}^{q}(p)+\frac{1}{m} d^{q}(p) .
$$

Obviously, for any $p \in P$,

$$
\begin{gather*}
w_{i}^{q}(p) \geqq \sum_{t} p(t) \omega_{i}(t) \\
\sum_{t} w_{i}^{q}(p)=\sum_{i} p(t) \omega(t)+\pi^{q}(p)  \tag{5}\\
w_{i}^{q}(p) \geqq \operatorname{Min}\left\{\sum_{t=1}^{T} p(t)\left(x_{i}(t)+\Delta v_{i}(t)\right)\left\{\left\{x_{i}(t), \Delta v_{i}(t)\right\}_{1}^{T} \in \mathscr{C}_{i}\right\} .\right. \tag{6}
\end{gather*}
$$

The first inequality follows from (d.5) and (1) above, and the last inequality from (c). Note that $Y$ is a polar of $P$, because of (2) and (4).

Now let $\zeta^{q}: \quad R^{l T} \rightarrow R^{l T}$ be a correspondence such that

$$
\begin{gather*}
\zeta^{q}(p) \equiv\left\{\left\{\sum_{i}\left(x_{i}(t)+\Delta v_{i}(t)\right)-\omega(t)-(b(t)-a(t))\right\}_{1}^{T} \in R^{l T} \mid\right. \\
\left.c_{i} \in \psi_{i}\left(p, w_{i}^{q}(p)\right), y \in \eta^{q}(p)\right\} \\
\zeta^{q} \text { is upper semicontinuous in } P \tag{7}
\end{gather*}
$$

since both $\psi_{i}$ and $\eta^{q}$ are upper semicontinuous in $P$, and closedness of a correspondence $f: X \rightarrow Y$, together with continuity of a single mapping $g: Y \rightarrow Z$, insures closedness of $g_{\circ} f: X \rightarrow Z$, where $Y$ is compact.

$$
\begin{align*}
& \zeta^{q}(p) \text { is convex for and } p \in P . \\
& p \zeta^{q}(p) \leqq 0 \quad \text { for any } p \in P, \tag{8}
\end{align*}
$$

because of (5) and (6).

$$
\begin{equation*}
\zeta^{q}(p) \neq \phi \quad \text { for all } p \in P \tag{9}
\end{equation*}
$$

because of (6). Since $Y$ is a polar of $P$, since $\zeta^{q}(p)$ is always in some compact set because of the compactness of $\mathscr{C}_{i}$ and $\mathscr{Y}^{q}$, and since (3), (7), (8), (9) hold, we can apply the theorem in Debreu [2] to our case, to obtain the existence of $p^{q} \in P$ such that

$$
Y \cap \zeta^{q}\left(p^{q}\right) \neq \phi .
$$

If we let $z$ be any point in the intersection,
there exists $y \in A \mathscr{Y}$ such that $z=\{b(t)-a(t)\}_{1}^{T}$, and there exist $c_{i}^{q} \in \psi_{i}\left(p^{q}, w_{i}^{q}\left(p^{q}\right)\right)$ and $\bar{y}^{q} \in \eta^{q}\left(p^{q}\right)$ such that

$$
z \equiv\left\{\sum_{i}\left(x_{i}^{q}(t)+\Delta v_{i}^{q}(t)\right)-\omega(t)-\left(\bar{b}^{q}(t)-\bar{a}^{q}(t)\right)\right\}_{1}^{T} .
$$

Define

$$
y^{q} \equiv y+\bar{y}^{q}
$$

and adopting the similar technique to that in Debreu [4], we see,

$$
\begin{gathered}
p^{q} \in P \\
y^{q} \in \mathscr{Y}, \\
c_{i}^{q} \in \mathscr{C}_{i}, \\
\sum_{i}\left(x_{i}^{q}(t)+\Delta v_{i}^{q}(t)\right)-\omega(t)-\left(b^{q}(t)-a^{q}(t)\right)=0, \\
c_{i}^{q} \in \psi_{i}\left(p^{q}, w_{i}^{q}\left(p^{q}\right)\right), \\
w_{i}^{q}\left(p^{q}\right)=p^{q} \omega_{i}+\sum_{j} \theta_{i j} \pi_{j}^{q}\left(p^{q}\right)+\frac{1}{m} d^{q}\left(p^{q}\right) \\
\sum_{t=1}^{T} p^{q}(t)\left(x_{i}^{q}(t)+\Delta v_{i}^{q}(t)\right)=w_{i}^{q}\left(p^{q}\right) \\
\sum_{t=1}^{T} p^{q}(t)\left(b^{q}(t)-a^{q}(t)\right)=\pi^{q}\left(p^{q}\right)
\end{gathered}
$$

Finally, the fact that $\left(p^{q},\left(c_{i}^{q}\right), y^{q}\right)$ lies in a compact set $P \times\left(\hat{\mathscr{C}}_{i}\right) \times \hat{\mathscr{Y}}$ for any $q$ enables us to find a convergent subsequence, the limit of which turns out the required quasi-equilibrium point.
Q.E.D.

## VII. Generalizations

So far, we have restricted ourselves to the consideration of a special type of economy in which neither wear and tear of goods nor the factors of uncertainty played an important role. The purpose of this section is to generalize the model so as to take these factors into consideration. We shall first formally present a new model, and then discuss its economic interpretations.

## Model:

Our new model differs from the preceding one mainly in two points.
The first point is that, while there were $l$ kinds of commodities for each period in the economy of the preceding model, we now admit that the number of the kinds of commodities at a given period depends upon the period when they exist, so that there are $l(t)$ kinds of commodities at the $t$ th period, where $l$ is now a function of $t$.

The second point is that we now assume that if a consumer saves $\alpha_{i}(t)\left(\in R^{l(t)}\right)$ units of commodities at the $t$ th period, they will turn into $\beta_{i}(t+1) \equiv F_{t i}\left(\alpha_{i}(t)\right)$ ( $\in R^{l(t+1)}$ ) units of commodities at the beginning of the next period, where $F_{t i}: \Omega^{l(t)} \rightarrow \Omega^{l(t+1)}$ might be called a transformation function. The concept $V$ can be used no longer, although its essence will still be retained. The new "tableau économique" might be Fig. II. We shall assume for any $i$ and $t$,


FIGURE II
(*) A single valued function $F_{t i}: \Omega^{l(t)} \rightarrow \Omega^{l(t+1)}$ is everywhere continuous, $\left(^{* *}\right) \quad F_{t i}(0)=0$.
As a consequence, $\mathscr{C}_{i}\left(\in R^{2[l(1)+l(2)+\cdots+l(T)]}\right)$ is a set of all meaningful $\left\{x_{i}(t)\right.$, $\left.\alpha_{i}(t)\right\}_{1}^{T}, \gtrsim_{i}$ is defined on the new $\mathscr{C}_{i}$, and the behavior of the $i$ th consumer can be summarized as,
maximizeing his preference,
subject to $\left\{x_{i}(t), \alpha_{i}(t)\right\}_{1}^{T} \in \mathscr{C}_{i}$,

$$
\begin{aligned}
\sum_{t=1}^{T} p(t)\left(x_{i}(t)\right. & \left.+\alpha_{i}(t)-F_{t-1, i}\left(\alpha_{i}(t-1)\right)\right) \\
& \leqq \sum_{t=1}^{T} p(t)\left(\omega_{i}(t)+\sum_{j} \theta_{i j}\left(b_{j}(t)-a_{j}(t)\right)\right)
\end{aligned}
$$

given $p(t), \omega_{i}(t), a_{j}(t), b_{j}(t) \in R^{l(t)}$, and

$$
0 \equiv F_{0 i}\left(\alpha_{i}(0)\right) \in R^{l(1)}, \quad 0 \equiv b_{j}(1), \quad t=1, \ldots, T, \quad j=1, \ldots, n
$$

Production sets, producer's behavior and others must also be understood mutatis mutandis.

The conditions $\left(^{*}\right)\left({ }^{* *}\right)$, together with the variants of (a.1)-(d.5), ensure the existence of a quasi-equilibrium. Indeed, one can easily prove it, following the techniques in the proof of Lemmas 3, 4 and Theorem.

Economic Interpretation (I):
As we have noted, commodities were not distinguished according to their ages in the preceding model, so that only a special type of physical depreciation (e.g., diminishing-balance method) could be treated there. We shall remove this restriction with the aid of the transformation function $F_{t i}: \Omega^{l(t)} \rightarrow \Omega^{l(t+1)}$ and the production set $\mathscr{\mathscr { H }}_{t j} \subset \Omega^{l(t)+l(t+1)}$. The amount $\alpha_{i, h(\tau)}(t)$ of the commodity, "the $h$ th good which is $\tau$ years old", will be transformed into the amount $\beta_{i, h(\tau+1)}(t+1)$ of another commodity, "the $h$ th good which is $(\tau+1)$ years old", (where the shape of the function $F_{t i}$ makes allowences for wear and tear of the good, of course), and the same to capital goods on production side. See von Neumann [9].

Economic Interpretation (II):
We now consider the factors of uncertainty within the framework of our new model. Following Debreu [3] Ch. 7, we define events at any future period, that can be conveniently represented by the verteces of a tree, the number of which is assumed finite. ${ }^{(2)}$ The $t$ th period has supposedly $k_{t}$ kinds of events, where by an event $e_{t} \in\left\{1_{t}, 2_{t}, \ldots, k_{t}\right\}$ atmospheric conditions, natural disasters, technical possibilities. . . are described for all the preceding periods, in addition to those of the $t$ th period.

A commodity can now be defined by its event (or vertex of the event tree as is drawn in Fig. III, for example), as well as by its physical characteristics, its location and its age.

(2) Radner's definition in his recent paper [11] are not useful in our case. It is in this sence that Debreu's definition of events is convenient when we treat stocks and flows of goods.

Future contracts are made at the beginning of the first period: The claim for any future good in the case of realization of an event is determined in the markets. See Debreu [3] 7. 3. in detail.

The transformation function $F_{t i}$ also plays an important role here. Suppose the $i$ th consumer saves in the $t$ th period the $h\left(\tau, 1_{t}\right)$ th commodity as much as $\alpha_{i h(\tau, 1 t)}$ unit(s), which should be understood as the amount of "the claim for the $h$ th good that is $\tau$ years old, in the case of realization of the $1_{t}$ th event". Then, it will be transformed into the amount $\beta_{i, h\left(\tau+1, e_{t+1}\right.}(t+1)$ of all the commodities "the claim for the $h$ th good that is $(\tau+1)$ years old, in the case of realization of the $e_{t+1}$ th event", where $e_{t+1}$ 's are compatible with $1_{t}$, that is, $e_{t+1}$ describes the same history up to the $t$ th as $1_{t}$. In the example of Fig. III, if a consumer saves $\alpha_{i, h\left(\tau, 2_{2}\right)}(2)$ unit(s) of the $h\left(\tau, 2_{2}\right)$ th commodity in the second period, and if there is no depreciation, then he will get in the next period the $h\left(\tau+1,4_{3}\right)$ th commodity and the $h\left(\tau+1,5_{3}\right)$ th, where

$$
\beta_{i, k\left(\tau+1,4_{3}\right)}(3)=\beta_{i, h\left(\tau+1,5_{3}\right)}(3)=\alpha_{i, h\left(\tau, 2_{2}\right)}(2) .
$$

The amount of the commodities $h\left(\tau+1,1_{3}\right), h\left(\tau+1,2_{3}\right), h\left(\tau+1,3_{3}\right)$, $h\left(\tau+1,6_{3}\right), h\left(\tau+1,7_{3}\right), h\left(\tau+1,8_{3}\right)$ might be well determined independent of the amount of the $h\left(\tau, 2_{2}\right)$ th.

All the interpretations written above should be applied also to the production side, mutatis mutandis.

In closing this section, I should like to retrieve a theorem in my survey article [6]:

Let $V_{i}=\sum_{s=1}^{S} \pi_{i s} U_{i}$, where $\left(\pi_{i 1}, \ldots, \pi_{i s}\right)$ is a probability vector. (Here, we are using the same notations as in Arrow [1].) Then,
(i) $U_{i}$ is quasi-concave if $V_{i}$ is quasi-concave.
(ii) $U_{i}$ is concave if and only if $V_{i}$ is concave.

It turns out that, if we adopt the Expected-Utility Hypothesis rigorously analyzed first by von Neumann \& Morgenstern [10], then the character of diversifier in Arrow's sense ensures concavity of his preference function, hence the condition (b.3) in the present paper.

## REFERENCES

[1] Arrow, K. J., "Le rôle des valeurs boursières pour la repartition la meilleure des risque," Econométrie (Centre National de la Recherche Scientifique, Paris 1953), translated in English as "The Role of Securities in the Optimal Allocation of Risk-bearing," Review of Economic Studies, Vol. 31 (2) (April, 1964).
[2] Debreu, G., "Market Equilibrium,'" Proceedings of the National Academy of Sciences of the U.S.A., Vol. 42, No. 11 (November, 1956).
[3] Debreu, G., Theory of Value (John Wiley and Sons, New York 1959).
[4] Debreu, G., "New Concepts and Techniques for Equilibrium Analysis," International Economic Review, Vol. 3, No. 3 (September, 1962).
[5] Gale, D., "The Closed Linear Model of Production," Paper 18 from Linear Inequalities and Related Systems, ed. by Kuhn and Tuker, Annals of Mathematical Studies 38 (Princeton, 1956).
[6] Ichiishi, T., "Uncertainty and Equilibrium Analysis," Mita Gakkai Zasshi, Vol. 62, No. 3 (March, 1969) (in Japanese).
[7] Malinvaud, E., "Capital Accumulation and Efficient Allocation of Resources," Econometrica, Vol. 21, No. 2 (April, 1953).
[8] Malinvaud, E., "Efficient Capital Accumulation: A Corrigendum," Econometrica, Vol. 30, No. 3 (July, 1962).
[9] von Neumann, J., "Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes," Ergebnisse eines mathematischen Kolloquiums, Vol. 8 (1937), translated in English as "A Model of General Economic Equilibrium," Review of Economic Studies, Vol. 13 (1945-46).
[10] von Neumann, J. and Morgenstern, O., Theory of Games and Economic Behavior, 2nd. ed. (Princeton Univ. Press, Princeton 1947).
[11] Radner, R., 'Competitive Equilibrium under Uncertainty,' Econometrica, Vol. 36, No. 1 (January, 1968).


[^0]:    (1) Our model in this context is inferior to von Neumann's growth model [9], because we do not distinguish commodities according to age, so that only a special type of depreciation can be treated. The case which is no less general than that in von Neumann [9] will be treated in Section VII.

