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TECHNICAL PROGRESS AND THE SURROGATE PRODUCTION FUNCTION

TAKAHIRO MIYAO

I. INTRODUCTION

The theory of capital is one of the most controversial subjects of contemporary economics. It is widely known that there are two opposing concepts of capital in the theory of economic growth. While one (the fixed-proportions school) supposes the capital good which has a technologically fixed labor requirement, the other (the neoclassical school) imagines one which is smoothly substitutable for labor. In order to synthesize these two concepts, Johansen [1] recently presented new model which incorporated elements of both. In his model—so called the "Johansen vintage model"—new machines can be built with any labor intensity, but existing machines, once produced, must be combined with labor in fixed proportions.

Solow [3], who especially developed the capital theoretic aspect of the two-sector version of the Johansen vintage model, demonstrated that "even in the absence of expost substitutability between labor and capital, the neoclassical categories of thought make sense and even the neo-classical theorems continue to hold". Furthermore, he [4] concentrated upon the representation of a heterogeneous-capital fixed-coefficient (ex-post) technology by a homogeneous-capital smooth production function, and concluded that the aggregate Cobb-Douglas production function which represents the long-run interrelationship between inputs and output, is useful in the estimation of distributive shares and the marginal productivity of "capital". Thānh [5] pointed out, however, that the validity of Solow's conclusions depends crucially on his special assumption that there is no possibility of technical change. If embodied technical progress is assumed so that even in the long-run (on the balanced growth path) the heterogeneity of capital can be preserved, the usefulness of the 'Surrogate' production function is very limited in the calculation of relative shares and the rate of return.

In this paper we shall try to show that, although, as Thānh pointed out, the usefulness of the surrogate production function is very limited in general, it is such a production function which gives the right answer in the estimation of distributive shares and marginal productivities under an interesting and important condition.

II. THE MODEL

We consider an economy with two sectors. In the first sector, consumption good is produced by machines and labor. Machines are assumed to be made by labor alone in the second sector. While consumption good is homogeneous, machines are heterogeneous, each differing in terms of the lifetime and the labor intensity which cannot be varied after their installation. Without loss of generality, we can choose machine-units in such a way that a unit of machine has a capacity of one unit of consumption goods at the time of construction. A unit of machine has a fixed labor intensity throughout its lifetime.

We assume that the total cost (the amount of labor) required to produce a unit of machine depends upon its labor intensity and its lifetime:

$$C(t\,\lambda\,N) = e^{-\gamma t}\lambda(t)^{-\alpha}N(t)^{\beta} \qquad 0<\alpha,\ \beta<1,\ \gamma \geqq 0$$

where λ (t) and N (t) denote the labor intensity and the lifetime of machine produced at time t, respectively and γ is the rate of "neutral" technical progress in the capital goods sector. The output rate at time t, $Q_\tau(t)$ of all machines produced at time τ is given by

$$Q(t) = \begin{cases} 0 & \text{if } t-\tau > N(\tau) \text{ and/or } \lambda(\tau)w(t) > e^{\delta(t-\tau)} \\ e^{\delta(t-\tau)}I(\tau) & \text{otherwise} \end{cases}$$

where, $w(t)$ is the wage rate (in terms of the consumption goods) at time t and $I(t)$ denotes the investment (in terms of machine unit) at time t, and δ is the rate of "neutral" technical progress in the consumption sector. Then, the present value of the expected net revenue (total discounted quasi-rent) over the lifetime of a new machine invested at time t, may be expressed as

$$\int_t^{t+N(t)} (e^{\delta(\tau-t)} - \lambda(t)w(\tau))e^{-\int_t^\tau \rho(u)du}d\tau - w(t)e^{-\gamma t}\lambda(t)^{-\alpha}N(t)^{-\beta}$$

where $\rho(t)$ is the rate of interest at time t. If entrepreneurs are assumed to expect that the interest rate will remain constant, while the wage rate will rise at a constant rate σ, the above expression reduces to

$$\int_t^{t+N(t)} e^{-(\rho-\delta)(\tau-t)}d\tau - \int_t^{t+N(t)} \lambda(t)w(t)e^{-(\rho-\sigma)(\tau-t)}d\tau - w(t)e^{-\gamma t}\lambda(t)^{-\alpha}N(t)^{\beta}$$

Then enterpreneurs' maximizing behavior and perfect competition lead to the following conditions.

(1) $\varphi(\rho-\sigma\ N) = \alpha e^{-\gamma t}\lambda^{-(1+\alpha)}N^{\beta}$

where $\varphi(x\,N) \equiv \int_0^N e^{-x\tau}d\tau$

(2) $e^{-(\rho-\delta)N} - \lambda(t)w(t)e^{-(\rho-\sigma)N} = w(t)\beta e^{-\gamma t}\lambda^{-\alpha}N^{\beta-1}$

(3) $\varphi(\rho-\delta\ N) - \lambda(t)w(t)\varphi(\rho-\sigma\ N) = w(t)e^{-\gamma t}\lambda^{-\alpha}N^{\beta}$

The total output of consumption goods at time t is

(4) $Q(t) = \int_{-\infty}^t Q_\tau(t)d\tau = \int_{\tau\in T(t)} e^{\delta(t-\tau)}I(\tau)d\tau$

where

$$T(t) = \{\tau|\ \min(e^{\delta(t-\tau)} - \lambda(\tau)w(t),\ N(\tau)-(t-\tau)) \geqq 0\}$$

The condition of full employment of labor can be written as

$$(5)\quad \int_{\tau \in T(t)} \lambda(\tau) I(\tau) d\tau + e^{-\gamma t}\lambda^{-\alpha} N^{\beta} I(t) = L(t)$$

where $L(t)$ denotes the supply of labor which is given exogenously. Finally, the equality of investment and saving may be expressed as

$$(6)\quad w(t)e^{-\gamma t}\lambda^{-\alpha}N^{\beta}I(t) = s\left\{\int_{\tau \in T(t)} e^{\delta(t-\tau)} I(\tau)d\tau + w(t)e^{-\gamma t}\lambda^{-\alpha}N^{\beta}I(t)\right\}$$

where $0 < s < 1$.

Given a past history, (1)–(6) can determine the equilibrium values of $\lambda(t)$, $N(t)$, $w(t)$, $\rho(t)$, $I(t)$ and $Q(t)$.

III. BALANCED GROWTH

We assume that the total supply of labor grows at the constant rate n:

$$L(t) = L_0 e^{nt}$$

Then, we can find a balanced growth solution of the system (1)–(6) as follows

$$(1)^*\quad \varphi(\rho - \sigma N(t)) = \alpha e^{-\gamma t}\lambda(t)^{-(1+\alpha)}N(t)^{\beta}$$

$$(2)^*\quad e^{-(\rho-\delta)N} - \lambda(t)w(t)e^{-(\rho-\sigma)N} = \lambda(t)w(t)\beta e^{-\gamma t}\lambda(t)^{-(1+\alpha)}N(t)^{\beta-1}$$

$$(3)^*\quad \varphi(\rho - \delta N(t)) - \lambda(t)w(t)\varphi(\rho - \sigma N(t)) = \lambda(t)w(t)e^{-\gamma t}\lambda(t)^{-(1+\alpha)}N(t)^{\beta}$$

where,

$$\lambda(t) = \lambda e^{-(\gamma/(1+\alpha))t}$$

$$w(t) = we^{(\gamma/(1+\alpha))t}$$

$$N(t) = N$$

$$\sigma = \frac{\gamma}{1+\alpha}$$

$$(4)^*\quad Q(t) = \int_{t-N}^{t} e^{\delta(t-\tau)} I(\tau)d\tau$$

$$(5)^*\quad \int_{t-N}^{t} \lambda(\tau)I(\tau)d\tau + e^{-\gamma t}\lambda(t)^{-\alpha}N(t)^{\beta}I(t) = L_0 e^{nt}$$

$$(6)^*\quad w(t)e^{-\gamma t}\lambda(t)^{-\alpha}N(t)^{\beta}I(t) = s\left\{\int_{t-N}^{t} e^{\delta(t-\tau)}I(\tau)d\tau + w(t)e^{-\gamma t}\lambda(t)^{-\alpha}N(t)^{\beta}I(t)\right\}$$

where

$$I(t) \equiv Ie^{gt} = Ie^{\{n+(\gamma/(1+\alpha))\}t} \quad \left(\text{or } \frac{\dot{I}(t)}{I(t)} = g = n + \frac{\gamma}{1+\alpha}\right)$$

or

$$(4)^{**}\quad Q(t) = I(t)\varphi(g - \delta N)$$

$$(5)^{**}\quad \left\{\lambda(t)\varphi\left(g - \frac{\gamma}{1+\alpha} N\right) + e^{-rt}\lambda(t)^{-\alpha}N(t)^{\beta}\right\}I(t) = L_0 e^{nt}$$

$$(6)^{**}\quad w(t)e^{-rt}\lambda(t)^{-\alpha}N(t)^{\beta}I(t) = s\{I(t)\varphi(g - \delta\ N) + w(t)e^{-rt}\lambda(t)^{-\alpha}N(t)^{\beta}I(t)\}$$

where, the equilibrium (stationary) value of ρ and the initial values λ, w, N and I are given by solving the following equations.

$$(7)\quad \varphi\left(\rho - \frac{\gamma}{1+\alpha} N\right) = \alpha\lambda^{-(1+\alpha)}N^{\beta}$$

$$(8)\quad e^{-(\rho-\delta)N} - \lambda w e^{-\{\rho-(\gamma/(1+\alpha))\}N} = \lambda w\beta\lambda^{-(1+\alpha)}N^{\beta}$$

$$(9)\quad \varphi(\rho - \delta\ N) - \lambda w\varphi\left(\rho - \frac{\gamma}{1+\alpha} N\right) = \lambda w\lambda^{-(1+\alpha)}N^{\beta}$$

$$(10)\quad \left\{\varphi\left(g - \frac{\gamma}{1+\alpha} N\right) + \lambda^{-(1+\alpha)}N^{\beta}\right\}\lambda I = L_0$$

$$(11)\quad \lambda w\lambda^{-(1+\alpha)}N^{\beta} = s\{\varphi(g - \delta\ N) + \lambda w\lambda^{-(1+\alpha)}N^{\beta}\}$$

It should be pointed out that the following relations hold on the balanced growth path.

I *The equilibrium rate of interest equals the balanced growth rate of investment if and only if* $s = 1/(2 + \alpha)$.

⟨proof⟩
From (7) and (9), we obtain

$$(12)\quad \lambda w = \frac{\alpha}{1+\alpha}\frac{\varphi(\rho - \delta)}{\varphi\left(\rho - \frac{\gamma}{1+\alpha}\right)}$$

which, together with (9) and (11), reduces to

$$\frac{\varphi(\rho - \delta\ N)}{\varphi(g - \delta\ N)} = \frac{s(1+\alpha)}{s}$$

This implies that the relation $\rho = g$ holds if and only if $s = 1/(2+\alpha)$ or $s(1+\alpha)/(1-s)=1$, because the expression φ is a monotone-descreasing function with respect to ρ.

II *Then, the saving ratio is equal to the relative share of profit:*

$$s = \pi \equiv \frac{Y(t) - w(t)L(t)}{Y(t)}\left(= \frac{Y(0) - wL_0}{Y(0)}\right)$$

where

$$Y(t) \equiv Q(t) + w(t)\lambda(t)^{-\alpha}N(t)^{\beta}I(t)$$

⟨proof⟩
From (10) (11) and the definition of $Q(t)$, we have

$$Q(0)\left[\lambda w\frac{\varphi\left(g - \frac{\gamma}{1+\alpha}\right)}{\varphi(g - \delta)} + \frac{s}{1-s}\right] = wL_0$$

and from (12), we obtain

$$Q(0)\left[\frac{\alpha}{1+\alpha}\frac{\varphi(\rho-\delta)}{\varphi\left(\rho-\frac{\gamma}{1+\alpha}\right)}\frac{\varphi\left(g-\frac{\gamma}{1+\alpha}\right)}{\varphi(g-\delta)}+\frac{s}{1-s}\right]=wL_0$$

Thus, $s=1/(2+\alpha)$ and $\rho=g$ imply $Q(0)=wL$, which, in turn, gives us the relation

$$s=\frac{Y(0)-Q(0)}{Y(0)}=\frac{Y(0)-wL_0}{Y(0)}=\pi$$

VI. THE SURROGATE PRODUCTION FUNCTION

We have shown that the value of λ (t) on the balanced growth path falls continuously at a constant rate $\gamma/(1+\alpha)$, because of the existence of technical progress in the capital goods sector. Even on the balanced growth path, therefore, hetorogeneity of capital will still be preserved. Now we try to answer to the question whether a homogeneous-capital smooth production function could be found to imitate closely the production process with heterogeneous-capital and to obtain the correct estimation of distributive shares and the rate of return.

Firstly, the stock of capital may be measured by its historical undepreciated cost in terms of labor time;

$$K(t)\equiv\int_{t-N}^{t}e^{-\gamma\tau}\lambda(\tau)^{-\alpha}N^{\beta}I(\tau)d\tau=e^{-\gamma\tau}\lambda(t)^{-\alpha}N^{\beta}I(t)\varphi\left(g-\frac{\gamma}{1+\alpha}N\right)$$

The stock of capital, defined above, is growing at a constant rate $g-(1/(1+\alpha))$, while $Y(t)$ and $Q(t)$ are growing at the same rate as investment, $I(t)=Ie^{gt}$. Thus, $K(t)/Q(t)$ or $K(t)/Y(t)$ is still falling, even on the balanced growth path, at a constant rate $\gamma/1+\alpha$. Such a state of affairs should not be called 'balanced' growth.

Next, we define the stock of capital by its historical undepreciated cost in terms of consumption goods;

$$K(t)\equiv\int_{t-N}^{t}w(\tau)e^{-\gamma\tau}\lambda(\tau)^{-\alpha}N^{\beta}I(\tau)d\tau=w(t)e^{-\gamma t}\lambda(t)^{-\alpha}N^{\beta}I(t)\varphi(g\,N)$$

At this time, it is easy to see that $K(t)$ is growing at the same rate as $Y(t)$ and $Q(t)$.

Then, the surrogate production function in the economy as a whole can be formulated as

(13) $Y(t)=Be^{(\gamma/(2+\alpha))t}K(t)^{1/(2+\alpha)}L(t)^{1-(1/(2+\alpha))}$

where

$$B=B(\rho N)\equiv\left(\frac{1+\alpha}{\alpha}\right)^{1/(2+\alpha)}N^{-(1/(2+\alpha))}$$

$$\times\left(\frac{\varphi\left(\rho-\frac{\gamma}{1+\alpha}N\right)}{\varphi(\rho-\delta N)}\cdot\frac{\varphi\left(g-\frac{\gamma}{1+\alpha}N\right)+\frac{1}{\alpha}\varphi\left(\rho-\frac{\gamma}{1+\alpha}N\right)}{\varphi(gN)}\right)^{1/2+\alpha}$$

$$\times\frac{\varphi(g-\delta N)+\frac{\varphi(\rho-\delta N)}{1+\alpha}}{\varphi\left(g-\frac{\gamma}{1+\alpha}N\right)+\frac{1}{\alpha}\varphi\left(\rho-\frac{\gamma}{1+\alpha}N\right)}.$$

⟨proof⟩

From (1)*, (4)**, (5)** and the definition of $Y(t)$ we have

$$\frac{1}{\lambda(t)}=\frac{Y(t)}{L(t)}\frac{\varphi\left(g-\frac{\gamma}{1+\alpha}N\right)+\frac{1}{\alpha}\varphi\left(\rho-\frac{\gamma}{1+\alpha}N\right)}{\varphi(g-\delta N)+\frac{1}{1+\alpha}\varphi(\rho-\delta N)}.$$

Thus

$$K(t)=\frac{1}{\lambda(t)}\frac{\alpha}{1+\alpha}\frac{\varphi(\rho-\delta N)}{\varphi\left(\rho-\frac{\gamma}{1+\alpha}N\right)}\cdot e^{-rt}\lambda(t)^{-\alpha}N^{\beta}\frac{L(t)}{\lambda(t)}$$

$$\times\frac{\varphi(gN)}{\varphi\left(g-\frac{\gamma}{1+\alpha}N\right)+\frac{1}{\alpha}\varphi\left(\rho-\frac{\gamma}{1+\alpha}N\right)}$$

$$=\frac{\alpha}{1+\alpha}N^{\beta}\frac{\varphi(\rho-\delta N)}{\varphi\left(\rho-\frac{\gamma}{1+\alpha}N\right)}\frac{\varphi(gN)}{\varphi\left(g-\frac{\gamma}{1+\alpha}N\right)+\frac{1}{\alpha}\varphi\left(\rho-\frac{\gamma}{1+\alpha}N\right)}$$

$$\times\left(\frac{\varphi\left(g-\frac{\gamma}{1+\alpha}N\right)+\frac{1}{\alpha}\varphi\left(\rho-\frac{\gamma}{1+\alpha}N\right)}{\varphi(g-\delta N)+\frac{1}{1+\alpha}\varphi(\rho-\delta N)}\right)^{2+\alpha}\cdot e^{-rt}\left(\frac{Y(t)}{L(t)}\right)^{2+\alpha}\cdot L(t)$$

$$=B^{-(2+\alpha)}e^{-rt}\left(\frac{Y(t)}{L(t)}\right)^{2+\alpha}L(t)$$

V. RELATIVE SHARES AND THE RATE OF RETURN

We can calculate the relative share of capital in aggregate output as

$$\pi\equiv\frac{Y(t)-w(t)L(t)}{Y(t)}=\frac{\int_{t-N}^{t}(e^{\delta(t-\tau)}-\lambda(\tau)w(t))I(\tau)d\tau}{\int_{t-N}^{t}e^{\delta(t-\tau)}I(\tau)d\tau+w(t)e^{-rt}\lambda(t)^{-\alpha}N^{\beta}I(t)}$$

$$=\frac{\varphi(g-\delta N)-\lambda(t)w(t)\varphi\left(g-\frac{\gamma}{1+\alpha}N\right)}{\varphi(g-\delta N)+\lambda(t)w(t)\frac{1}{\alpha}\varphi\left(\rho-\frac{\gamma}{1+\alpha}N\right)}$$

$$= \frac{1 - \dfrac{\alpha}{1+\alpha} \dfrac{\varphi(\rho - \delta\, N)}{\varphi\left(\rho - \dfrac{\gamma}{1+\alpha}\, N\right)} \dfrac{\varphi\left(g - \dfrac{\gamma}{1+\alpha}\, N\right)}{\varphi(g - \delta\, N)}}{1 + \dfrac{1}{1+\alpha} \dfrac{\varphi(\rho - \delta\, N)}{\varphi(g - \delta\, N)}}$$

In general, this result cannot be obtained from the surrogate production function (13). It should be pointed out, however, that if ρ is equal to g, then

$$\pi = \frac{1 - \dfrac{\alpha}{1+\alpha}}{1 + \dfrac{1}{1+\alpha}} = \frac{1}{2+\alpha} \quad \text{(which can also be give by [II])}$$

and the surrogate production function can provide us the correct value of relative shares. Considering that $\rho = g$ if and only if $s = 1/(2+\alpha)$, we may conclude that *when the saving ratio is equal to the elasticity of aggregate output with respect to capital in the surrogate production function* (*or when the equilibrium rate of interest is equal to the balanced growth rate of investment*) *the relative share of capital is also equal to the elasticity of output to labor in the surrogate production function.*

Finally, we shall investigate the calculation of the rate of return or the marginal productivity of capital. On the balanced growth path, the rate of return r may be expected constant and the equality between the discounted gross return of capital and its cost in terms of consumption goods will hold.

$$\int_{t-N}^{t} r e^{-\rho\tau} d\tau = 1 \qquad \text{or} \qquad r = \frac{1}{\varphi(\rho\, N)}$$

In general, this result also cannot be obtained from the surrogate production function, which only provides us the following information

$$\frac{1}{2+\alpha} \frac{Y(t)}{K(t)} = \frac{1}{2+\alpha} \frac{\varphi(g - \delta\, N) + w(t) e^{-rt} \lambda(t)^{-\alpha} N^{\beta}}{w(t) e^{-rt} \lambda(t)^{-\alpha} N^{\beta} \varphi(g\, N)}$$

$$= \frac{1}{2+\alpha} \frac{1 + \dfrac{1}{1+\alpha} \dfrac{\varphi(\rho - \delta\, N)}{\varphi(g - \delta\, N)}}{\dfrac{1}{1+\alpha} \dfrac{\varphi(\rho - \delta\, N)}{\varphi(g - \delta\, N)} \cdot \varphi(g\, N)}$$

It is easy to see, however, that the above expression becomes exactly equal to r if $\rho = g$:

$$\frac{1}{2+\alpha} \frac{Y(t)}{K(t)} = \frac{1}{2+\alpha} \frac{2+\alpha}{\varphi(g\, N)} = \frac{1}{\varphi(\rho\, N)} = r$$

Thus, when $s = 1/(2+\alpha)$ (or $\rho = g$), the surrogate production function also gives us the correct value in the estimation of the rate of return or the marginal productivity of capital.

VI. CONCLUSION

We have shown that the stock of capital measured by historical cost in terms of consumption goods is more appropriate than that measured in terms of historical undepreciated labor requirement, and though, in general, the usefulness of the surrogate production function is very limited, distributive shares and the rate of return can be correctly calculated from such a production function under the condition that the saving ratio equals the elasticity of output to capital or the equilibrium rate of interest equals the balanced growth rate of investment;

$$s\left(=\frac{1}{2+\alpha}\right)=\pi \quad \text{or} \quad \rho=g$$

It should be noted that this is the same condition as one for the Neo-classical Theorem or Maximum Consumption [2]. From such a normative viewpoint, our surrogate production function may be regarded as the representation of some *optimal* relation between output and inputs in the long run: if per capita consumption is to be maximized, the relative share of capital must actually be equal to the elasticity of output to capital in the surrogate production function. But it would require further investigation that goes beyond the limits of this short piece to prove the interesting assertion.

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