A Thesis for the Degree of Ph.D. in Science

Metric Theory of Diophantine Approximations over Imaginary Quadratic Fields

March 2016

Graduate School of Science and Technology

Keio University

Zhengyu Chen

Preface

The present thesis discusses the metric theory of Diophantine approximations for complex numbers. In 1941, R. J. Duffin and A. C. Schaeffer made a conjecture on metric theory of Diophantine approximations. As one of partial results, in 1978, J. D.Vaaler proved this conjecture under an additional condition. For the exceptional sets of Diophantine approximations, i.e., Lebesgue measures of the sets are 0, we use Hausdorff dimension to measure their size. G. Harman generalized the result of V. Jarník and A. S. Besicovitch and proved that the Hausdorff dimension of the set which satisfies the Duffin-Schaeffer conjecture is 1. In this thesis, we discuss the metric theory of Diophantine approximation over an imaginary quadratic field and show that a Vaaler type theorem holds in this case. Also we extend G. Harman's results to the imaginary quadratic fields.

Chapter 1 gives some results on metric theory of Diophantine approximations for real numbers and complex numbers. In chapter 2, we extend the Duffin-Schaeffer conjecture to the imaginary quadratic fields and gives our result about the Vaaler type theorem over imaginary quadratic fields and its proof. Then in chapter 3, we discuss Jarník and Besicovitch's result over imaginary quadratic fields and give the Hausdorff dimension of the set of Duffin-Schaeffer conjecture over imaginary quadratic fields without the co-prime condition.

Standing notation. We denote by $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$, the set of real numbers, rational numbers, integers, and strictly positive integers, respectively.

Contents

Introduction

Diophantine approximations deal with the problems of aproximations of real numbers by using rational numbers. The first result of Diophantine approximations is given by Dirichlet. Since then, many other interesting results have been given by many other mathematicians. In the earlier of the 20th century, A. Y. Khintchine considered a metric theory of Diophantine approximation problem, i.e., with what conditions a non-negative real valued function $\psi(n)$ on $\mathbb{R}$ should be satisfied such that the inequality

$$\left|\alpha - \frac{m}{n}\right| < \frac{\psi(n)}{n}\ ,\ (m,n) = 1, \tag{1}$$

has infinitely many solutions of positive integers m and n for almost all real numbers α. Here $(m,n) = 1$ denotes that m and n are co-prime, and the word "almost all" means that the set of real numbers $\alpha \in [0,1)$ has full Lebesgue measure, i.e., in real number case it is 1. Inversely, we use the word "almost no" to denote Lebesgue measure 0. Then in 1941, R. J. Duffin and A. C. Schaeffer made a conjecture on a metric theory of Diophantine approximation problem in their paper [7]. The conjecture states that the inequality (1) has infinitely many solutions of positive integers m and n for almost all real numbers α if and only if $\sum_{n=1}^{\infty} \varphi(n)\psi(n)n^{-1} = \infty$. Here $\varphi(n)$ is Euler function which counts the positive integers less than or equal to n that are relatively prime to n. If $\sum_{n=1}^{\infty} \varphi(n)\psi(n)n^{-1}$ converges, then we can easily see that the inequality (1) has only finitely many solutions of positive integers m and n for almost all α. So the only difficulty is proving the inequality (1) has infinitely many solutions for almost all α under the condition $\sum_{n=1}^{\infty} \varphi(n)\psi(n)n^{-1} = \infty$. R. J. Duffin and A. C. Schaeffer also gave a sufficient condition on $\psi(n)$ for having infinitely many solutions a.e., which is called the Duffin-Schaeffer theorem. In 1950, J. W. S. Cassels [5] showed that the inequality $|\alpha - m/n| < \psi(n)/n$ without the condition of $(m,n) = 1$, has infinitely many solutions for either almost all α or almost no α. Then in 1961, P. X. Gallagher [10] added the condition of $(m,n)=1$ on and gave the conclusion that, the inequality (1) has infinitely many solutions for either almost all α or almost no α. In 1970, P. Erdös [8] showed that if $\psi(n)=0$ or εn^{-1} for all $n\in\mathbb{N}$ and some $\varepsilon>0$, then the inequality (1) has infinitely many solutions of positive integers m and n for almost all α whenever $\sum_{n=1}^{\infty}\varphi(n)\psi(n)n^{-1}$ diverges. In 1978, J. D. Vaaler [28] gave a more general result following P. Erdös' idea. More precisely, he proved that the inequality (1) has infinitely many solutions of positive integers m and n for almost all α, if $\psi(n)=\mathcal{O}(n^{-1})$ and $\sum_{n=1}^{\infty}\varphi(n)\psi(n)n^{-1}$ diverges.

In metric theory of Diophantine approximations, we also need to measure the size of the sets of α with Lebesgue measure 0. In this case, we usually use the Hausdorff dimension to measure the size of the exceptional sets instead of the Lebesgue measure, since the Hausdorff dimension of the sets can be not 0 even if their Lebesgue measures are all 0. In 1929, V. Jarnik [16] proved that the Hausdorff dimension of the set of $\alpha\in\mathbb{R}$ such that the inequality

$$\left|\alpha-\frac{m}{n}\right|<\frac{1}{n^{\gamma}}$$

has infinitely many solutions of rational numbers m/n is $2/\gamma$ for $\gamma>2$, and also in 1934 A. S. Besicovitch [3] proved the same result. G. Harman [11] then showed a more general result that the Hausdorff dimension of the set of $\alpha\in\mathbb{R}$ such that the inequality $|n\alpha-m|<n^{-\rho}$ with $(m,n)=1$ and $\gamma=\sup\{0\leqslant h:\sum_{n\in\mathcal{A}}n^{-h}\text{ diverges}\}$ for some infinite set $\mathcal{A}$ of positive integers has infinitely many solutions of rational numbers m/n equals to $(1+\gamma)/(1+\rho)$. We note that V. Jarnik and A. S. Besicovitch's results can be followed as its corollary. G. Harman also proved that the Hausdorff dimension of the set of real numbers which have infinitely many solutions to the Diophantine inequality concerning the Duffin-Schaeffer conjecture [7] is 1 by using this result.

Diophantine approximations for complex numbers was first considered in 1887-88 by A. Hurwitz [14], who discussed the Diophantine approximation problem by continued fractions over the imaginary quadratic fields $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$. Since then, a number of papers discussed this subject such as [9], [22] and [20]. In 1982, D.Sullivan [26] gave a metric result of Diophantine approximation over an imaginary quadratic field under a condition similar to the condition of the Duffin-Schaeffer theorem. In 1991, H. Nakada and G. Wagner proved a Duffin-Schaeffer type theorem over an imaginary quadratic field as well as a Gallagher type theorem [21]. In this thesis, we discuss a further development of the metric theory of Diophantine approximations over an imaginary quadratic field. Our main result indicates that the difficulty of the complex number version of the Duffin-Schaeffer conjecture is similar to that of the one dimensional real case. Indeed, we will show that a Vaaler type theorem holds in this case and then we find the same difficulty in case of real numbers for proving the complex version of the Duffin-Schaeffer conjecture.

For a given square-free negative integer d, we consider

$$\mathbb{Q}(\sqrt{d}) = \left\{ p + q\sqrt{d} : p, q \in \mathbb{Q} \right\}$$

and its maximal order $\mathbb{Z}[\omega]$, i.e.

$$\mathbb{Z}[\omega] = \{ m + n\omega : m, n \in \mathbb{Z} \},$$

where

$$\omega = \begin{cases} (1+\sqrt{d})/2, & \text{if } d \equiv 1 \pmod 4 \\ \sqrt{d}, & \text{if } d \equiv 2, 3 \pmod 4). \end{cases}$$

Define the set of fundamental area

$$\mathbb{F} = \{ z : z = x + y\omega,\ x, y \in \mathbb{R},\ 0 \le x, y < 1 \}, \tag{2}$$

which is a subset of $\mathbb{C}$. In order to avoid the problem of different prime factor decompositions of an integer in $\mathbb{Z}[\omega]$, we consider ideals for the uniqueness of the prime factor decomposition. For an integer $a \in \mathbb{Z}[\omega]$, we denote by (a) the principal ideal generated by a. Then we can give a complex number version of the Duffin-Schaeffer conjecture as follows: suppose $\Psi((r))$ is a non-negative real valued function defined on the set of principal ideals of $\mathbb{Z}[\omega]$, then the inequality

$$\left| z - \frac{a}{r} \right| < \frac{\Psi((r))}{|r|}\ ,\ (r, a) = (1), \tag{3}$$

has infinitely many solutions r and a with $r, a \in \mathbb{Z}[\omega]$ for almost all $z \in \mathbb{C}$ if and only if $\sum \Phi((r))\Psi^2((r))|r|^{-2} = \infty$. Here (r, a) denotes the ideal in $\mathbb{Z}[\omega]$ generated by r and a, and $(r, a) = (1)$ denotes that r and a are co-prime in terms of ideals. $\Phi((r))$ is the Euler functions over imaginary quadratic fields which counts the reduced residue classes modulo (r) and it also counts the integers over imaginary quadratic fields in the fundamental area $\mathbb{F}$ that are relatively prime to r. Without loss of generality, we discuss our problems for almost all $z \in \mathbb{F}$ instead of $z \in \mathbb{C}$.

As one of the main result of the thesis, we show that the Vaaler type theorem for complex numbers holds, that is, if $\Psi((r)) = \mathcal{O}(|r|^{-1})$, then the Duffin-Schaeffer conjecture over imaginary quadratic fields is true. It is different from the real number case, the difficulty of dealing this theorem over imaginary quadratic fields is that there are many different ideals with the same norm. In real number case, the positive number itself denotes the distance between the positive number and origin. However, this doesn't make sense for imaginary quadratic fields case. So it makes difficulties when we consider the convergence of the sum of norm of ideals and sieve method of ideals. In the last section of chapter 2, by following an idea of R. J. Duffin and A. C. Schaeffer [7], we show an example by giving a sequence of $\Psi((r))$ whose sum diverges but the measure of the set of the Duffin-Schaeffer conjecture over imaginary quadratic fields under our choice of $\{\Psi((r))\}$ is less than 1. This shows that the convergence condition of the Duffin-Schaeffer conjecture over imaginary quadratic fields is reasonable.

In this thesis, we also show the generalized Jarník and Besicovitch's theorem over the imaginary quadratic fields and prove that the Hausdorff dimension of the set of complex numbers satisfy the Duffin-Schaeffer conjecture over imaginary quadratic fields without the co-prime condition is 2, by using this result.

Chapter 1

Background

1.1 On metric theory of Diophantine approximations for real numbers

1.1.1 Diophantine approximations

The problem of Diophantine approximation is the approximation to irrational numbers by rational numbers, and what we are interested in is that we want to know how the irrational numbers can be approximated by rational numbers. The earliest result about this problem is given by Dirichlet in 1842, see [25].

Given a real number α, let $[\alpha]$, the integer part of α, denote the greatest integer smaller than or equals to α, and let $\{\alpha\} = \alpha - [\alpha]$. Then $\{\alpha\}$ is the fractional part of α, and satisfies $0 \leqslant \{\alpha\} < 1$. Also, let $\|\alpha\|$ denote the distance from α to the nearest integer. Then always $0 \leqslant \|\alpha\| \leqslant 1/2$.

Theorem 1.1.1 (Dirichlet (1842))**.** *Let α and Q be real numbers with $Q > 1$. Then there exist integers m, n such that $1 \leqslant n < Q$ and $|\alpha n - m| \leqslant \frac{1}{Q}$.*

It follows from Theorem 1.1.1 that if α is an irrational number, there are infinitely many fractions m/n in lowest terms with

$$\left|\alpha - \frac{m}{n}\right| \leqslant \frac{1}{Qn} < \frac{1}{n^2},$$

and we have the following corollary.

Corollary 1.1.2. *Suppose that α is an irrational number. Then there exist infinitely many pairs m, n of relatively prime integers with*

$$\left|\alpha - \frac{m}{n}\right| < \frac{1}{n^2}. \tag{1.1}$$

Note that this corollary is not true if α is a rational number. For suppose that $\alpha = u/v$ with $u, v \in \mathbb{Z}$. If $\alpha \neq m/n$, then

$$\left|\alpha - \frac{m}{n}\right| = \left|\frac{u}{v} - \frac{m}{n}\right| = \left|\frac{nu - mv}{vn}\right| \geqslant \frac{1}{vn},$$

and therefore (1.1) can be satisfied by only finitely many pairs m, n of relatively prime integers.

In fact, for almost all $\alpha \in \mathbb{R}$, the inequality

$$\left|\alpha - \frac{m}{n}\right| < \frac{1}{2n^2}$$

has infinitely many solutions m, n with $(m, n) = 1$. It holds from the continued fractions expansion of α as follows:

Theorem 1.1.3 (Vahlen (1895))**.** *Let $p_{\ell-1}/q_{\ell-1}, p_\ell/q_\ell$ be consecutive convergents to α. Then at least one of them satisfies*

$$\left|\alpha - \frac{m}{n}\right| < \frac{1}{2n^2}.$$

Hurwitz [15] improved this further to $\frac{1}{\sqrt{5}}n^{-2}$.

Theorem 1.1.4 (Hurwitz (1891))**.** *(i) For every irrational number α there are infinitely many distinct rationals m/n with*

$$\left|\alpha - \frac{m}{n}\right| < \frac{1}{\sqrt{5}n^2}.$$

(ii) This would be wrong if $\sqrt{5}$ were replaced by a constant $A > \sqrt{5}$.

This is best possible, as can be seen by considering $\alpha = (-1 + \sqrt{5})/2$. We have $\alpha^2 + \alpha - 1 = 0$, so for any fraction m/n:

$$|m/n - \alpha| = \left|\frac{(m/n)^2 + m/n - 1}{m/n + \beta}\right| \geqslant \frac{1}{n^2}\frac{1}{m/n + \beta}, \tag{1.2}$$

where $\beta = (1 + \sqrt{5})/2$. Now if $|m/n - \alpha| < n^{-2}$, then by (1.2) we have

$$|m/n - \alpha| \geqslant \frac{1}{n^2(\sqrt{5} + n^{-2})}.$$

Hence the factor $\sqrt{5}$ is best possible.

It was shown by E. Borel [4] and F. Bernstein [2] in the earlier of 20th century that almost all real numbers have unbounded partial quotients in their continued fraction expansion, hence for irrational α we have

$$\left|\alpha - \frac{p_\ell}{q_\ell}\right| < \frac{1}{q_\ell q_{\ell+1}},$$

where p_ℓ/q_ℓ is the ℓth convergent to α. Thus for almost all $\alpha \in \mathbb{R}$, there are infinitely many fractions m/n with $|\alpha - m/n| = o(n^{-2})$. A. Y. Khintchine [17] used the results of E. Borel and F. Bernstein in conjunction with an estimate for the growth of q_ℓ to prove the following very precise result.

1.1.2 On metric theory of Diophantine approximations

For some given approximation inequality, we want to know the size of the sets of all α with the approximation inequality which has infinitely many solutions m/n. This is the metric problem of Diophantine approximations. A. Y. Khinchine made an interesting result as follows:

Theorem 1.1.5 (Khintchine (1924)). *Let $\psi(n)$ be a positive continuous function. Then the inequality*

$$\left|\alpha - \frac{m}{n}\right| < \frac{\psi(n)}{n} \tag{1.3}$$

(i) has infinitely many solutions in integers $m, n > 0$ for almost all real numbers α if

$$\sum_{n=1}^{\infty} \psi(n) \tag{1.4}$$

diverges and $n\psi(n)$ is non-increasing. (ii) has at most finitely many solutions in integers $m, n > 0$ for almost all real numbers α if the sum (1.4) converges.

Clearly, if (1.3) has infinitely many solutions, there will be infinitely many fractions m/n in lowest terms satisfying (1.3), since $x\psi(x)$ is non-increasing. It follows that almost all real numbers have infinitely many approximations of the form $|\alpha - m/n| < (n^2 \log n)^{-1}$, but only finitely many of the form $|\alpha - m/n| < n^{-2}(\log n)^{-1-\varepsilon}$ for any $\varepsilon > 0$. Of course, this says nothing about approximating to α uniformly as in Dirichlet's theorem.

From Khinchine's theorem, we consider the conditions of the non-negative real valued function ψ such that the inequality

$$\left|\alpha - \frac{m}{n}\right| < \frac{\psi(n)}{n}$$

has infinitely many solutions m, n for almost all α. We can see a simple necessary condition

$$\sum_{n=1}^{\infty} \varphi(n)\frac{\psi(n)}{n} = \infty,$$

since we have

Theorem 1.1.6. *Let $\psi(n)$ be a non-negative-valued function such that*

$$\sum_{n=1}^{\infty} \psi(n)\frac{\varphi(n)}{n} \tag{1.5}$$

converges. Then there are only a finite number of solutions to the inequality

$$\left|\alpha - \frac{m}{n}\right| < \frac{\psi(n)}{n}, \quad (m,n) = 1, \quad n \geqslant 1 \tag{1.6}$$

for almost all $\alpha \in \mathbb{R}$.

This is easy to prove if we use the first Borel-Cantelli lemma.

Lemma 1.1.7 (the first Borel-Cantelli lemma)**.** *Let X be a measure space with measure μ. Let $A_j (j = 1, 2, 3, ...)$ be a collection of measurable subsets of X. If*

$$\sum_{n=1}^{\infty} \mu(A_j) < \infty,$$

then almost all members of X (with respect to μ) belong to only finitely many of the A_j.

Consider $\alpha \in [0, 1)$ without lose of generality. Let

$$\varepsilon_n = [0,1) \bigcap \left(\bigcup_{\substack{m=1 \\ (m,n)=1}}^{n-1} \left(\frac{m - \psi(n)}{n}, \frac{m + \psi(n)}{n} \right) \right)$$

for all n. So if (1.6) has infinitely many solutions for some $\alpha \in [0, 1)$, then $\alpha \in \cup_{m=1}^{\infty} \cap_{n=1}^{m} \varepsilon_n$. However, by the inequality

$$\sum_{n=1}^{\infty} \lambda(\varepsilon_n) \leqslant 2 \sum_{n=1}^{\infty} \varphi(n)\frac{\psi(n)}{n}$$

and the first Borel-Cantelli lemma we see that for almost all $\alpha \in [0, 1)$, it only belongs to finitely many of ε_n, which means Theorem 1.1.6 holds.

In higher dimensions Khintchine extended his theorem to simultaneous approximation, and clearly Theorem 1.1.6 can be extended almost immediately.

In 1941, Duffin and Schaeffer [7] made the following conjecture which provoked much research and remains to date one of the most important unsolved problems in metric number theory.

Conjecture 1 (Duffin-Schaeffer conjecture (1941))**.** *Let $\psi(n)$ be a non-negative-valued function such that the sum (1.5) diverges. Then (1.6) has infinitely many solutions for almost all $\alpha \in \mathbb{R}$.*

Duffin and Schaeffer also constructed a function $\psi(n)$ to show that if we use (1.4) instead of (1.5) in Duffin-Schaeffer conjecture, then the result of Duffin-Schaeffer conjecture does not hold. Thus, the condition of divergence of (1.3) can not guarantee the existence of infinitely many solutions to (1.6). So the divergence of (1.5) is reasonable for Duffin-Schaeffer conjecture.

Although the Duffin-Schaeffer conjecture has not been proved yet, the k-dimensional Duffin-Schaeffer conjecture has been proved by A. D. Pollington and R. C. Vaughan [23].

Theorem 1.1.8 (Pollington and Vaughan (1989))**.** *Let $k > 1$ and let $\{\beta_n\}$ denote a sequence of real numbers with*

$$0 \leqslant \beta_n < \frac{1}{2}$$

and suppose that

$$\sum_{n=1}^{\infty} \left(\frac{\beta_n \varphi(n)}{n} \right)^k$$

diverges. Then the inequalities

$$\max(|\alpha_1 n - a_1|, ..., |\alpha_k n - a_k|) < \beta_n, \quad (a_i, n) = 1, \quad i = 1, ..., k,$$

have infinitely many solutions for almost all $\alpha = (\alpha_1, \alpha_2, ..., \alpha_k) \in \mathbb{R}^k$.

Note that the theorem holds when $k > 1$, and we can not use it for the case $k = 1$ to show that the Duffin-Schaeffer conjecture is true.

P. X. Gallagher [10] and J. W. S. Cassels [5] gave the following result:

Theorem 1.1.9. *(A).(Cassels (1950)) Let $\psi(n)$ be a sequence of non-negative reals. Then the inequality*

$$|\alpha n - m| < \psi(n) \tag{1.7}$$

has infinitely many solutions for either almost all α or almost no α.
(B).(Gallagher (1961)) The conclusion of part (A) holds with the additional condition $(m,n) = 1$ imposed in (1.7).

The theorem shows that zero-one laws operate in both the problem of approximation by all fractions and the problem of approximation by reduced fractions.

In 1970, P. Erdös [8] proved the Duffin-Schaeffer conjecture to be true when, for some $\varepsilon > 0$, $\psi(n)$ takes on only the values 0 or ε/n.

Theorem 1.1.10 (Erdös (1970))**.** *The Duffin-Schaeffer conjecture is true when there exists a subset $A \subset \mathbb{N}$ such that if*

$$\sum_{n\in A} \frac{\phi(n)}{n^2} = \infty, \tag{1.8}$$

then for almost all $\alpha \in \mathbb{R}$ the inequality $|\alpha - m/n| < \varepsilon/n$ for $n \in A$ has infinitely many solutions m/n.

J. D. Vaaler [28] modified the P. Erdös' method to obtain a more general result as follows:

Theorem 1.1.11 (Vaaler (1978))**.** *The Duffin-Schaeffer conjecture is true when $\psi(n) = \mathcal{O}(1/n)$.*

Vaaler's result is, so far, the best sufficient simple condition. We can see that the theorem of Khintchine, 1.1.5, is a special case of Vaaler's result. So the hard case of proving Duffin-Schaeffer conjecture is if $\sum \phi(n)\psi(n)n^{-1} = \infty$ and $\psi(n)$ oscilating hardly.

Catlin made a conjecture and showed that it is equivalent to the Duffin-Schaeffer conjecture. However, Vaaler also pointed out that Catlin's proof of the equivalence of two conjectures contains a serious flaw, see [28].

Recently, V. Beresnevich, G. Harman, A. Haynes and S. Velani [1] [13] gave an divergent condition

$$\sum_{n\geqslant 16} \frac{\varphi(n)\psi(n)}{n \cdot \exp(c(\log\log n)(\log\log\log n))} = \infty \quad for\ some\ c > 0$$

which is equivalent to the condition of the divergence of (1.5).

1.1.3 Hausdorff dimension of the exceptional sets

For some set of real numbers whose Lebesgue measure is positive, its Hausdorff dimension is 1. So for sets whose Lebesgue measure is 0, it is meaningful by using Hausdorff dimension to measure the size of the sets.

Definition 1 (Hausdorff dimension)**.** *We denote by* $|\cdot|$ *the length of an interval. The Hausdorff dimension of a set of real numbers* φ *is* d*, i.e,* $\dim_H \varphi = d$ *if it satisfies the following 2 conditions:*
(i) For any $\beta > d$ *and any* $\varepsilon > 0$*, there exists a sequence of intervals* $\{\mathcal{I}_j\}$ *such that*

$$\varphi \subset \bigcup_{j=1}^{\infty} \mathcal{I}_j, \ \sum_{j=1}^{\infty} |\mathcal{I}_j|^{\beta} < 1, \ |\mathcal{I}_j| < \varepsilon \text{ for all } j.$$

(ii) For any $\beta < d$*, there exists* $\varepsilon > 0$ *such that there exists no sequence of intervals which satisfies all the three conditions above.*

V. Jarník, in 1929, and A. S. Besicovitch, in 1934, considered the Hausdorff dimension of the exceptional sets of real numbers such that the inequality has infinitely many solutions with $\sum \psi(n) < \infty$. Thus we see that the Lebesgue measure of these exceptional sets is 0.

Theorem 1.1.12 (Jarník(1929), Besicovitch(1934))**.** *If* $\gamma > 2$*, then the Hausdorff dimension of the set of* $\alpha \in \mathbb{R}$ *such that the inequality*

$$\left|\alpha - \frac{m}{n}\right| < \frac{1}{n^{\gamma}}$$

has infinitely many solutions m, n *is* $2/\gamma$*.*

G. Harman generalized Jarník and Besicovitch's result as follows:

Theorem 1.1.13 (generalized Jarník and Besicovitch's theorem(by Harman, 1998))**.** *For an infinite subset* $\mathcal{A}$ *of* $\mathbb{N}$*, let*

$$\gamma = \sup\left\{h \geqslant 0 : \sum_{n \in \mathcal{A}} n^{-h} = \infty\right\}.$$

For a real number ρ *with* $\rho > \gamma$*, then the Hausdorff dimension of the set of* $\alpha \in \mathbb{R}$ *such that the inequality*

$$|n\alpha - m| < n^{-\rho}, \ (m, n) = 1, \ n \in \mathcal{A}$$

has infinitely many solutions m, n *is* $\frac{1+\gamma}{1+\rho}$*.*

We see that the result of Jarník and Besicovitch is a corollary to Theorem (1.1.13). Harman then estimated the Hausdorff dimension of the set of real numbers which satisfy the properties in the statement of the Duffin-Schaeffer conjecture by using the generalized Jarník and Besicovitch's theorem, and proved that its Hausdorff dimension is 1.

Theorem 1.1.14 (Harman, 1998)**.** *The Hausdorff dimension of the set of real numbers which satisfy the Duffin-Schaeffer conjecture is* 1*.*

1.2 On metric theory of Diophantine approximations for complex numbers

1.2.1 Diophantine approximations for complex numbers

A. Hurwitz [14] introduced, in 1887, continued fraction expansions for complex numbers with Gaussian integers as partial quotients, via the nearest integer algorithm, known as Hurwitz algorithm, and established some basic properties concerning convergence of the sequence of convergents, and also proved an analogue of the classical Lagrange theorem characterizing quadratic surds as the numbers with eventually periodic continued fractions; analogous results were also proved for the nearest integer algorithms with respect to Eisenstein integers as partial quotients, in place of Gaussian integers. Then in the earlier of the 20th century, L. R. Ford [9] and O. Perron [22] also did some studies about these problems.

1.2.2 On metric theory of Diophantine approximations for complex numbers

D. Sullivan [26], in 1983, and H. Nakada [20], in 1990, showed some results about the metric theory of Diophantine approximations for complex numbers. Then in 1991, H. Nakada and G. Wagner [21] showed Gallagher's 0-1 laws over the complex numbers, that is, either the set of complex numbers satisfying Duffin-Schaeffer conjecture or its complement is a set of Lebesgue measure 0 even if

$$\sum_{r\in\mathbb{Z}[\omega]\setminus\{0\}} \Psi^2((r)) = \infty. \tag{1.9}$$

That is

Theorem 1.2.1 (Nakada, Wagner, 1991)**.** $\lambda(\cdot)$ *denotes Lebesgue measure. Consider the inequality*

$$\left|z - \frac{a}{r}\right| < \frac{f(r)}{|r|}, (a, r) = 1, a, r \in \mathbb{Z}[\omega], \tag{1.10}$$

where f is a non-negative function defined on $\mathbb{Z}[\omega]$ with $f(r) = f(u \cdot r)$ for all units u in $\mathbb{Z}[\omega]$. Let A_f be the set of $z (\in \mathbb{F})$, for which (1.10) has infinitely many solutions. Then we have

$$\lambda(A_f) = 0 \text{ or } 1$$

for any non-negative function f, where λ denotes the normalized Lebesgue measure on $\mathbb{F}$.

By using this theorem, they proved a complex Duffin-Schaeffer theorem:

Theorem 1.2.2 (Nakada, Wagner, 1991)**.** *Suppose that*

$$\sum_{r \in \mathbb{Z}[\omega]} f^2(r) = \infty$$

and there exist infinitely many $R \in \mathbb{N}$ such that

$$\sum_{|r|<R, r \in \mathbb{Z}[\omega]} f^2(r) < c_1 \cdot \sum_{|r|<R, r \in \mathbb{Z}[\omega]} f^2(r) \cdot \varphi(r)/|r|^2$$

for some constant $c_1 > 0$. Then (1.10) has infinitely many solutions for almost all $z \in \mathbb{F}$.

If $\sum_{r \in \mathbb{Z}[\omega] \setminus \{0\}} \varphi((r)) \Psi^2((r)) |r|^{-2} < \infty$, then the normalized Lebesgue measure of the set of complex numbers satisfying the properties in the statement of the Duffin-Schaeffer conjecture is 0 due to the Borel-Cantelli lemma. We can not ignore the possibility that the measure of the set of complex numbers satisfying the properties in the statement of the Duffin-Schaeffer conjecture without co-prime condition equals to 0 under the condition (1.9). In the last section of this paper, by following an idea of Duffin and Schaeffer [7], we construct a counter example by giving a sequence of $\Psi((r))$ which satisfies (1.9) but the measure of the set of complex numbers satisfying the properties in the statement of the Duffin-Schaeffer conjecture without co-prime condition under our choice of $\{\Psi((r))\}$ is not 1.

Chapter 2

Vaaler type theorem for complex numbers

2.1 Vaaler type theorem over imaginary quadratic fields

Throughout this thesis we will use $\mathrm{N}(\cdot)$ for the norm of an ideal over $\mathbb{Z}[\omega]$, and use P (and P_j) for the prime ideals. We also use $\Phi(\cdot)$ to denote the Euler function over imaginary quadratic fields. Let $\mathbb{F}$ be the fundamental area, see (2).

The differences when we consider the Diophantine approximations over imaginary quadratic fields are mainly two points:
(i) The prime factor decomposition of an integer of $\mathbb{Z}[\omega]$ is not unique. For example, integral number 6 in $\mathbb{Q}(\sqrt{-5})$ has two prime factor decompositions, i.e., $6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$, so it is difficult to decide whether two integers over imaginary quadratic fields are co-prime or not.
(ii) How to estimate the number of integers which are relatively prime to some given integer $r \in \mathbb{Z}[\omega]$.

For (i), it is easy if we consider our problem over ideals instead of over complex numbers directly, since the factor decomposition of ideals is unique. For (ii), we use the Euler function $\Phi((r))$ over $\mathbb{Z}[\omega]$ with $\Phi((r)) = |r|^2 \prod_{P|(r)}(1 - \mathrm{N}^{-1}(P))$, which counts the number of residue classes modulo the principal ideal (r). It is also equal to the number of integers $a \in \mathbb{Z}[\omega]$ that are relatively prime to r and $a/r \in \mathbb{F}$.

Thus it makes sense for us to give the Duffin-Schaeffer conjecture over imaginary quadratic fields as follows:

Conjecture 2 (Duffin-Schaeffer conjecture over imaginary quadratic fields). *Let $r, a \in \mathbb{Z}[\omega]$ and let $\Psi((r))$ be a non-negative real valued function on ideals. If*

$$\sum_{(r):\text{principal ideal}} \Phi((r))\frac{\Psi^2((r))}{|r|^2} = \infty,$$

then for almost all $z \in \mathbb{C}$, the inequality

$$\left|z - \frac{a}{r}\right| < \frac{\Psi((r))}{|r|},\ (r, a) = (1) \tag{2.1}$$

has infinitely many solutions r, a.

Our main result is the following

Theorem 2.1.1. *If $\Psi((r)) = \mathcal{O}(|r|^{-1})$, then the inequality of (2.1) has infinitely many solutions of r and a with $r, a \in \mathbb{Z}[\omega]$ for almost all $z \in \mathbb{C}$, whenever*

$$\sum \Phi((r))\Psi^2((r))|r|^{-2} = \infty.$$

We first define $\mathcal{E}_{(r)}$ as the set of complex number z which satisfies the inequality of (2.1) for a given $r \in \mathbb{Z}[\omega]$, i.e.

$$\mathcal{E}_{(r)} = \bigcup_{\substack{a\in\mathbb{Z}[\omega] \\ \frac{a}{r}\in\mathbb{F} \\ (a,r)=(1)}} \left\{z : \left|z - \frac{a}{r}\right| < \frac{\Psi((r))}{|r|}, z \in \mathbb{F}\right\}.$$

It is enough for proving Theorem 2.1.1 to show

$$\lambda\left(\bigcap_{N=1}^{\infty}\bigcup_{|r|^2=N}^{\infty}\mathcal{E}_{(r)}\right) = \lim_{N\to\infty}\lambda\left(\bigcup_{|r|^2=N}^{\infty}\mathcal{E}_{(r)}\right) = 1 \tag{2.2}$$

holds under the conditions of $\Psi((r)) = \mathcal{O}(|r|^{-1})$ and $\sum \Phi((r))\Psi^2((r))|r|^{-2} = \infty$. Here λ denotes the normalized Lebesgue measure on $\mathbb{F}$.

We extend two theorems of Vaaler [28] (Theorem 2 and 3) to the imaginary quadratic field as follows:

Theorem 2.1.2. *Suppose there exist an integer $k \geq 2$ and a real number $\eta > 0$ such that the following condition holds: every finite subset $\mathbf{Z}$ of $\{k, k+1, k+2, \cdots\}$ for which $0 \leqslant \Lambda(\mathbf{Z}) \leqslant \eta$ and*

$$\sum_{|r|^2\in\mathbf{Z}}\sum_{\substack{|s|^2\in\mathbf{Z} \\ (r)\neq(s)}} \lambda(\mathcal{E}_{(r)} \cap \mathcal{E}_{(s)}) \leqslant \Lambda(\mathbf{Z}) \tag{2.3}$$

hold with $\Lambda(\mathbf{Z}) = \sum_{|r|^2\in\mathbf{Z}} \lambda(\mathcal{E}_{(r)})$, then $\sum \Phi((r))\Psi^2((r))|r|^{-2} = \infty$ implies (2.2).

Theorem 2.1.3. *If $\Psi((r)) = \mathcal{O}(|r|^{-1})$, then there exists $\eta > 0$ such that if $\mathbf{Z}$ is a finite subset of $\{2, 3, 4, \cdots\}$ which satisfies $0 < \Lambda(\mathbf{Z}) \leqslant \eta$, then*

$$\sum_{\substack{|r|^2\in\mathbf{Z} \\ (r)\neq(s)}} \sum_{|s|^2\in\mathbf{Z}} \lambda(\mathcal{E}_{(r)} \cap \mathcal{E}_{(s)}) \ll (\Lambda(\mathbf{Z}))^2 \left(\ln\ln \frac{1}{\Lambda(\mathbf{Z})}\right)^2. \tag{2.4}$$

We note that (2.4), the conclusion of Theorem 2.1.3, is stronger than (2.3) since there exists a sufficiently large rational integer k such that

$$\Lambda(\mathbf{Z})(\ln\ln \Lambda(\mathbf{Z})^{-1})^2 < 1$$

with $\mathbf{Z} = \{k, k+1, k+2, ...\}$. In the next section, we will prove Theorem 2.1.3 and then prove Theorem 2.1.2 which completes the proof of Theorem 2.1.1. We note that we do need the condition $\Psi((r)) = \mathcal{O}(|r|^{-1})$ in the proof of Theorem 2.1.3 and do not need it in the proof of Theorem 2.1.2.

The idea of the proof of Theorem 2.1.3 is based on Renyi-Lamperti's Borel-Cantelli type lemma:

Lemma 2.1.4 (Renyi-Lamperti's Borel-Cantelli type lemma)**.** *$A_1, A_2, ...$ is a sequence of events and $P(\cdot)$ is the probability function with*

$$\sum_{n=1}^{\infty} P(A_n) = \infty.$$

Then for the event set $A = \{\omega :$ there are infinitely many $n \in \mathbb{N}$ which satisfies $\omega \in A_n\}$, we have

$$P(A) \geqslant \limsup_{N\to\infty} \frac{(\sum_{n=1}^{N} P(A_n))^2}{\sum_{m,n=1}^{N} P(A_n \cap A_m)}.$$

The case of $P(A) > 0$ is showed by 0-1 law, and Theorem 2.1.3 is the case of $P(A) = 1$.

2.2 Proof of some lemmas

Before we give the proof of our result, first we state some results in algebraic number theory which will be used later in our proof. See [24] and [19].

Theorem 2.2.1 (Mertens' 2nd theorem over algebraic number fields)**.**

$$\sum_{\mathrm{N}(P)\leqslant x} \frac{1}{\mathrm{N}(P)} = \ln\ln x + B_{\mathbb{K}} + \mathcal{O}\left(\frac{1}{\ln x}\right),$$

where $B_{\mathbb{K}}$ is a constant depending only on the algebraic number field $\mathbb{K}$.

Theorem 2.2.2 (Mertens' 3rd theorem over algebraic number fields)**.**

$$\prod_{\mathrm{N}(P)\leqslant x} \left(1 - \frac{1}{\mathrm{N}(P)}\right)^{-1} = e^{\gamma}\alpha_{\mathbb{K}} \ln x + \mathcal{O}(1),$$

where $\gamma > 0$ and $\alpha_{\mathbb{K}}$ are constants depending only on the algebraic number field $\mathbb{K}$.

Theorem 2.2.3 (Landau prime ideal theorem)**.** *The number of prime ideals of norm $\leqslant y$ is*

$$\pi(y) = \mathrm{Li}(y) + \mathcal{O}(ye^{-c_{\mathbb{K}}\sqrt{\ln y}}),$$

where $c_{\mathbb{K}}$ is a constant depending only on the algebraic number field $\mathbb{K}$ and $\mathrm{Li}(y) = \int_2^y (1/\ln t)dt$.

Now we give some lemmas similar to Vaaler's estimates [28]. We denote by $g(R)$ for an ideal R of $\mathbb{Z}[\omega]$ as the smallest positive integer v that satisfies

$$\sum_{\substack{P|R \\ \mathrm{N}(P)>v}} \frac{1}{\mathrm{N}(P)} < 1.$$

Then we have the following result:

Lemma 2.2.4. *For an ideal R of $\mathbb{Z}[\omega]$, if $g(R) = v$, then we have*

$$\prod_{\substack{P|R \\ \mathrm{N}(P)\leqslant v}} \left(1 - \frac{1}{\mathrm{N}(P)}\right) \ll \frac{\Phi(R)}{\mathrm{N}(R)} \quad \text{as } v \to \infty.$$

Proof. From the formula of Euler's function over ideals, we have

$$\Phi(R) = \mathrm{N}(R) \prod_{P|R} \left(1 - \frac{1}{\mathrm{N}(P)}\right).$$

Then

$$\begin{aligned}\prod_{\substack{P|R\\ \mathrm{N}(P)\leqslant v}}\left(1-\frac{1}{\mathrm{N}(P)}\right) &= \frac{\Phi(R)}{\mathrm{N}(R)}\prod_{\substack{P|R\\ \mathrm{N}(P)>v}}\left(1-\frac{1}{\mathrm{N}(P)}\right)^{-1}\\ &= \frac{\Phi(R)}{\mathrm{N}(R)}\exp\left\{\sum_{\substack{P|R\\ \mathrm{N}(P)>v}}\ln\left(1-\frac{1}{\mathrm{N}(P)}\right)^{-1}\right\}\\ &\leqslant \frac{\Phi(R)}{\mathrm{N}(R)}\exp\left\{\sum_{\substack{P|R\\ \mathrm{N}(P)>v}}\frac{1}{\mathrm{N}(P)}+\sum_{P}\sum_{j=2}^{\infty}\frac{1}{j(\mathrm{N}(P))^j}\right\}.\end{aligned}$$

Now we see

$$\begin{aligned}\sum_{P}\sum_{j=2}^{\infty}\frac{1}{j(\mathrm{N}(P))^j} &\leqslant \sum_{P}\sum_{j=2}^{\infty}\frac{1}{(\mathrm{N}(P))^j}\\ &\leqslant \sum_{P}\frac{1}{\mathrm{N}(P)\,(\mathrm{N}(P)-1)}<\sum_{S}\frac{1}{\mathrm{N}^2(S)}.\end{aligned}\tag{2.5}$$

Here $\sum_S$ is a sum over all ideals of $\mathbb{Z}[\omega]$. In order to show that the right side of (2.5) converges, we first estimate the number of ideals whose norm is less than or equal to a given rational integer N. Denote by $T(N)$ the number of ideals whose norm is less than or equal to the given rational integer N. By [12], there exists a constant $k(d)$ such that

$$\lim_{N\to\infty}\frac{T(N)}{N}=k(d),$$

which shows that $u_N=T(N)/N$ is bounded. Denote by T_i the number of ideals whose norm is equal to $i\in\mathbb{N}$. Then we have $T(N)=\sum_{i=1}^{N}T_i$. From $T_N=Nu_N-(N-1)u_{N-1}$, we have

$$\begin{aligned}\sum_{\mathrm{N}(S)=1}^{N}\frac{1}{\mathrm{N}^2(S)} &= \sum_{i=1}^{N}\frac{T_i}{i^2}\\ &= \frac{u_N}{N}+\left(\frac{1}{(N-1)^2}-\frac{1}{N^2}\right)(N-1)u_{N-1}+\cdots+\frac{3}{4}u_1\\ &= \frac{u_N}{N}+\frac{2N-1}{(N-1)N^2}u_{N-1}+\frac{2N-3}{(N-2)(N-1)^2}u_{N-2}+\cdots+\frac{3}{4}u_1\\ &< \frac{u_N}{N}+\frac{2}{N^2}u_{N-1}+\frac{2}{(N-1)^2}u_{N-2}+\cdots+\frac{2}{2^2}u_1\\ &\ll \sum_{i=1}^{N}\frac{1}{i^2}\quad \textit{as } N\to\infty.\end{aligned}$$

So we see that the right side of (2.5) converges which implies

$$\prod_{\substack{P|R \\ \mathrm{N}(P)\leqslant v}} \left(1 - \frac{1}{\mathrm{N}(P)}\right) \ll \frac{\Phi(R)}{\mathrm{N}(R)} \quad as\ v \to \infty.$$

This completes the proof of Lemma 2.2.4. □

We will give a corollary to Lemma 2.2.4 which we use later.

Corollary 2.2.5. *For an ideal R of $\mathbb{Z}[\omega]$, if $g(R) = v$, then we have*

$$1 \ll \frac{\Phi(R)}{\mathrm{N}(R)} \ln(1+v) \quad as\ v \to \infty.$$

Proof. From M. Rosen's results of the 3rd Mertens' theorem 2.2.2 on an algebraic number field and Lemma 2.2.4, we have

$$\begin{aligned} 1 &\ll \frac{\Phi(R)}{\mathrm{N}(R)} \prod_{\substack{P|R \\ \mathrm{N}(P)\leqslant v}} \left(1 - \frac{1}{\mathrm{N}(P)}\right)^{-1} \\ &\leq \frac{\Phi(R)}{\mathrm{N}(R)} \prod_{\mathrm{N}(P)\leqslant v} \left(1 - \frac{1}{\mathrm{N}(P)}\right)^{-1} \\ &\ll \frac{\Phi(R)}{\mathrm{N}(R)} \ln(1+v) \quad as\ v \to \infty. \end{aligned}$$

□

We define a collection $\mathcal{N}(\xi, x, v)$ of ideals of $\mathbb{Z}[\omega]$ by

$$\mathcal{N}(\xi, x, v) = \left\{ R : \sum_{\substack{P|R \\ \mathrm{N}(P)\geq v}} \frac{1}{\mathrm{N}(P)} \geq \xi\ , \mathrm{N}(R) \leqslant x \right\},$$

where $\xi > 0$, $x > 0$, $v > 0$. We denote by $\#\mathcal{N}(\xi, x, v)$ the number of ideals in $\mathcal{N}(\xi, x, v)$. Then we can extend Vaaler's estimate [28] to the complex number case as follows:

Lemma 2.2.6. *For any $\varepsilon > 0$, $\xi > 0$ and $x > 0$, we have*

$$\#\mathcal{N}(\xi, v, x) \ll \frac{x}{e^{v^{\beta(1-\varepsilon)}}} \quad as\ v \to \infty \quad with\ \beta = e^{\xi}. \tag{2.6}$$

Proof. Suppose $0 < \varepsilon < 1 - \frac{1}{e^{\xi}} = 1 - \frac{1}{\beta}$. It is enough to show Lemma 2.2.6 for such ε since the right side of (2.6) becomes larger if ε goes larger. Let $[v, w]$ be an interval with $w = v^{\beta(1-\frac{2}{3}\varepsilon)}$. Let $\{P_1, P_2, \cdots, P_M\}$ be the set of all prime ideals whose norms are in $[v, w]$ with $\mathrm{N}(P_1) \leqslant \mathrm{N}(P_2) \leqslant \cdots \leqslant \mathrm{N}(P_M)$. Let π be the prime-counting function in the sense of ideals of $\mathbb{Z}[\omega]$, i.e. $\pi(w)$ is the number of prime ideals whose norm is less than or equal to w. Then we see $M \geq \pi(w) - \pi(v)$. We have the equality

$$\frac{v^{\beta(1-\frac{2}{3}\varepsilon)}}{\frac{w}{\ln w} - \frac{v}{\ln v}} = \frac{\beta(1-\frac{\varepsilon}{3})\ln v}{v^{\frac{\beta\varepsilon}{3}} - \beta(1-\frac{\varepsilon}{3})v^{1-\beta(1-\frac{2}{3}\varepsilon)}}.$$

Since $\varepsilon < 1 - 1/\beta$, we have $1 - \beta(1-\frac{2}{3}\varepsilon) < 0$. Hence there exists an integer $v_0(\varepsilon, \xi) > 0$ such that $\frac{w}{\ln w} - \frac{v}{\ln v} \geq v^{\beta(1-\frac{2}{3}\varepsilon)}$ for any $v \geq v_0$ and we have

$$M \geq \pi(w) - \pi(v) \gg \frac{w}{\ln w} - \frac{v}{\ln v} \geq v^{\beta(1-\frac{2}{3}\varepsilon)} \quad as\ v \ \to \infty$$

by the prime ideal theorem.

Next, we divide all the ideals in $\mathcal{N}(\xi, x, v)$ into two classes.

<u>**Class 1**</u>. There are no less than M different prime ideal factors of ideal R and norms of these prime ideal factors are all in the interval of $[v, e^w]$.

Denote by N_1 the number of ideals in Class 1. By using the 2nd Mertens' theorem 2.2.1 on an algebraic number field, we see

$$\begin{aligned} N_1 &\ll x\frac{\left(\sum_{v \leqslant \mathrm{N}(P) \leqslant e^w} \frac{1}{\mathrm{N}(P)}\right)^M}{M!} \leqslant x\frac{\left(\sum_{\mathrm{N}(P) \leqslant e^w} \frac{1}{\mathrm{N}(P)}\right)^M}{M!} \\ &\ll x\frac{(\ln w)^M}{M!} \quad as\ v \to \infty. \end{aligned}$$

Here we note $w \leqslant M^2$. From Stirling's formula $n! = \sqrt{2\pi n}\,(n/e)^n\left(1 + \mathcal{O}\left(n^{-1}\right)\right)$, we have

$$\begin{aligned} x\frac{(\ln w)^M}{M!} &\ll \frac{2^M(e^{\ln\ln M})^M}{M!} \\ &\ll x\frac{2^M e^{M+M\ln\ln M}}{M^M\sqrt{2\pi M}} \\ &< \frac{x}{e^{M(\ln M - \ln\ln M - 2)}} \cdot \frac{1}{\sqrt{2\pi M}} \\ &\ll \frac{x}{e^M} \ll \frac{x}{e^{v^{\beta(1-\frac{2}{3}\varepsilon)}}} \quad as\ v \to \infty. \end{aligned} \tag{2.7}$$

<u>**Class 2**</u>. There are less than M different prime ideal factors of ideal R and norms of these prime ideal factors are all in the interval of $[v, e^w]$.

By using the 2nd Mertens' theorem 2.2.1 on an algebraic number field, we have

$$\begin{aligned}\sum_{j=1}^{M}\frac{1}{\mathrm{N}(P_j)} = \sum_{v\leqslant \mathrm{N}(P)\leqslant w}\frac{1}{\mathrm{N}(P)} &\ll \ln\ln w - \ln\ln v\\ &= \xi + \ln\left(1-\frac{\varepsilon}{3}\right)\\ &< \xi - \frac{\varepsilon}{3} \quad as\ v\to\infty.\end{aligned}$$

From

$$\sum_{\substack{P|R\\ \mathrm{N}(P)\geqslant v\geqslant g(R)}}\frac{1}{\mathrm{N}(P)} = \sum_{\substack{P|R\\ v\leqslant \mathrm{N}(P)\leqslant w}}\frac{1}{\mathrm{N}(P)} + \sum_{\substack{P|R\\ w<\mathrm{N}(P)\leqslant e^w}}\frac{1}{\mathrm{N}(P)} + \sum_{\substack{P|R\\ \mathrm{N}(P)>e^w}}\frac{1}{\mathrm{N}(P)} \geq \xi$$

and the condition of Class 2, we see

$$\sum_{\substack{P|R\\ v\leqslant \mathrm{N}(P)\leqslant w}}\frac{1}{\mathrm{N}(P)} + \sum_{\substack{P|R\\ w<\mathrm{N}(P)\leqslant e^w}}\frac{1}{\mathrm{N}(P)} \leqslant \sum_{v\leqslant \mathrm{N}(P)\leqslant w}\frac{1}{\mathrm{N}(P)} \ll \xi - \frac{\varepsilon}{3} \quad as\ v\to\infty.$$

So we have the estimate

$$\sum_{\substack{P|R\\ \mathrm{N}(P)>e^w}}\frac{1}{\mathrm{N}(P)} \gg \frac{\varepsilon}{3} \quad as\ v\to\infty.$$

The number of ideals R of Class 2 is less than $\sum_{\mathrm{N}(R)\leqslant x} 1$ and then we see

$$\begin{aligned}\sum_{\mathrm{N}(R)\leqslant x} 1 &\ll \sum_{\mathrm{N}(R)\leqslant x}\frac{3}{\varepsilon}\sum_{\substack{P|R\\ \mathrm{N}(P)>e^w}}\frac{1}{\mathrm{N}(P)}\\ &\ll \frac{1}{\varepsilon}\sum_{\mathrm{N}(P)>e^w}\frac{1}{\mathrm{N}(P)}\cdot\frac{x}{\mathrm{N}(P)}\\ &< \frac{x}{\varepsilon}\left(\frac{1}{(e^w)^2}+\frac{1}{e^w(e^w+1)}+\frac{1}{(e^w+1)(e^w+2)}+\cdots\right)\\ &\ll \frac{x}{\varepsilon}\cdot\frac{1}{e^w} \ll \frac{1}{\varepsilon}\cdot\frac{x}{e^{v^{\beta(1-\frac{2}{3}\varepsilon)}}} \quad as\ v\to\infty. \end{aligned}\tag{2.8}$$

The estimates (2.7) and (2.8) imply (2.6), which completes the proof of Lemma 2.2.6. □

We define two collections $\mathcal{A}_r(\xi, v)$ and $\mathcal{B}_r(\xi, v)$ of ideals for a fixed $r \in \mathbb{Z}[\omega]$ by

$$\mathcal{A}_r(\xi, v) = \left\{ A : A|(r),\quad \sum_{\substack{P|A\\ \mathrm{N}(P)\geq v\geq g((r))}}\frac{1}{\mathrm{N}(P)} \geq \xi \right\},$$

$$\mathcal{B}_r(\xi, v) = \left\{ B : B|(r), \sum_{\substack{P|B \\ \mathrm{N}(P) \geq v \geq g((r))}} \frac{1}{\mathrm{N}(P)} < \xi \right\},$$

for $\xi > 0$, $v > 0$.

Lemma 2.2.7. *For any $\varepsilon > 0$, $\xi > 0$, and $v \geq g((r))$,*

$$\sum_{A \in \mathcal{A}_r(\xi,v)} \frac{1}{\mathrm{N}(A)} \ll \frac{\ln(1 + g((r)))}{e^{v^{\beta(1-\varepsilon)}}} \quad \textit{as } v \to \infty \textit{ with } \beta = e^{\xi}.$$

Proof. Let $w = v^{\beta(1-\frac{\varepsilon}{3})}$, where $0 < \varepsilon < 1 - e^{-\xi} = 1 - \beta^{-1}$. Suppose there are M different prime ideals $P_1, P_2, \cdots, P_M$ whose norms are in $[v, w]$ with $\mathrm{N}(P_1) \leqslant \mathrm{N}(P_2) \leqslant \cdots \leqslant \mathrm{N}(P_M)$. Let $\mathcal{J}$ be the collection of M different prime ideals whose norms are all in $[v, \infty)$. Then from the proof of Lemma 2.2.6, we have

$$\sum_{P \in \mathcal{J}} \frac{1}{\mathrm{N}(P)} \leqslant \sum_{j=1}^{M} \frac{1}{\mathrm{N}(P_j)} \ll \xi - \frac{\varepsilon}{3} \quad \textit{as } v \to \infty.$$

Since for any $A \in \mathcal{A}_r(\xi, v)$, we see

$$\sum_{\substack{P|A \\ \mathrm{N}(P) \geq v \geq g((r))}} \frac{1}{\mathrm{N}(P)} \geq \xi.$$

This implies that for all large v, there are at least no less than M different prime ideal factors of A whose norms are all in $[v, \infty)$. Let $Q_1, Q_2, \cdots, Q_J$ be all different prime ideal factors of (r).

<u>**Case 1**</u>. If $J < M$.

From the discussion in the above, we see that $\mathcal{A}_r(\xi, v) = \emptyset$ for all large v, which means $\sum_{A \in \mathcal{A}_r(\xi,v)} \mathrm{N}^{-1}(A) = 0$.

<u>**Case 2**</u>. If $J \geq M$.

Since $v \geq g((r))$ and $\sum_{j=1}^{J} \mathrm{N}^{-1}(Q_j) < 1$, we see

$$\sum_{A \in \mathcal{A}_r(\xi,v)} \frac{1}{\mathrm{N}(A)} \leqslant \sum_{A|(r)} \frac{1}{\mathrm{N}(A)} \cdot \frac{\left(\sum_{j=1}^{J} \frac{1}{\mathrm{N}(Q_j)} \right)^M}{M!} < \left(\sum_{A|(r)} \frac{1}{\mathrm{N}(A)} \right) \frac{1}{M!}. \tag{2.9}$$

Suppose $(r) = Q_1^{\gamma_1} Q_2^{\gamma_2} \cdots Q_J^{\gamma_J}$ where $Q_1, Q_2, \cdots, Q_J$ are all different prime ideal factors of (r) and $\gamma_1, \gamma_2, \cdots, \gamma_J$ are positive integers. By Corollary 2.2.5, we have

$$\begin{aligned}
\sum_{A|(r)} \frac{1}{\mathrm{N}(A)} &= \left(1 + \frac{1}{\mathrm{N}(Q_1)} + \frac{1}{\mathrm{N}^2(Q_1)} + \cdots + \frac{1}{\mathrm{N}^{\gamma_1}(Q_1)}\right) \\
&\cdot \left(1 + \frac{1}{\mathrm{N}(Q_2)} + \frac{1}{\mathrm{N}^2(Q_2)} + \cdots + \frac{1}{\mathrm{N}^{\gamma_2}(Q_2)}\right) \\
&\cdot \ \cdots \left(1 + \frac{1}{\mathrm{N}(Q_J)} + \frac{1}{\mathrm{N}^2(Q_J)} + \cdots + \frac{1}{\mathrm{N}^{\gamma_J}(Q_J)}\right) \\
&\leqslant \prod_{\substack{Q|(r) \\ Q \text{ is prime ideal}}} \left(1 - \frac{1}{\mathrm{N}(Q)}\right)^{-1} \\
&= \frac{|r|^2}{\Phi((r))} \ll \ln(1 + g((r))) \quad \text{as } v \to \infty. \qquad (2.10)
\end{aligned}$$

From (2.7), (2.9), and (2.10), we have

$$\sum_{A \in \mathcal{A}_r(\xi, v)} \frac{1}{\mathrm{N}(A)} \ll \frac{\ln(1 + g((r)))}{e^{v^{\beta(1-\varepsilon)}}} \quad \textit{as } v \to \infty \textit{ with } \beta = e^{\xi}.$$

This completes the proof of Lemma 2.2.7. □

Lemma 2.2.8. *Suppose $(s), (r)$ are two principal ideals with $s, r \in \mathbb{Z}[\omega]$ and $U = (s, r)$. Then for $\varepsilon > 0$, $\xi > 0$, $x > 0$, $y \geq 2$, $v \geq g((r))$, we have*

$$\sum_{\substack{(s)^v \\ x\mathrm{N}(U) < |s|^2 < xy\mathrm{N}(U)}} \frac{1}{|s|^2} \ll \frac{\ln(1 + g((r))) \ln y}{e^{v^{\beta(1-\varepsilon)}}} \quad \textit{as } v \to \infty, \qquad (2.11)$$

and

$$\sum_{\substack{(s)^v \\ x\mathrm{N}^{-1}(U) < |s|^2 < xy\mathrm{N}^{-1}(U)}} \frac{1}{|s|^2} \ll \frac{\ln(1 + g((r))) \ln y}{e^{v^{\beta(1-\varepsilon)}}} \quad \textit{as } v \to \infty, \qquad (2.12)$$

with $\beta = e^{\xi}$. Here $\sum_{(s)^v}$ means the sum over (s) satisfying $g((s)) = v$.

Proof. The right sides of (2.11) and (2.12) are both independent of U and x. Thus, by choosing x properly, we see that (2.11) and (2.12) are equivalent. So we only need to prove (2.11). Let $(s) = US'$ and $(r) = UR'$, then we see

$$\begin{aligned}
\sum_{\substack{(s)^v \\ x\mathrm{N}(U)<|s|^2<xy\mathrm{N}(U)}} \frac{1}{|s|^2} &= \sum_{U|(r)} \sum_{\substack{(s)^v \\ (s,r)=U \\ x\mathrm{N}(U)<|s|^2<xy\mathrm{N}(U)}} \frac{1}{|s|^2} \\
&= \sum_{U\in\mathcal{A}_r(\frac{1}{2},v)} \sum_{\substack{(s)^v \\ (s,r)=U \\ x\mathrm{N}(U)<|s|^2<xy\mathrm{N}(U)}} \frac{1}{|s|^2} \\
&+ \sum_{U\in\mathcal{B}_r(\frac{1}{2},v)} \sum_{\substack{(s)^v \\ (s,r)=U \\ x\mathrm{N}(U)<|s|^2<xy\mathrm{N}(U)}} \frac{1}{|s|^2}.
\end{aligned} \tag{2.13}$$

Here we see

$$\begin{aligned}
\sum_{U\in\mathcal{A}_r(\frac{1}{2},v)} \sum_{\substack{(s)^v \\ (s,r)=U \\ x\mathrm{N}(U)<|s|^2<xy\mathrm{N}(U)}} \frac{1}{|s|^2} &= \sum_{U\in\mathcal{A}_r(\frac{1}{2},v)} \sum_{\substack{(US')^v \\ (s,r)=U \\ x<\mathrm{N}(S')<xy}} \frac{1}{\mathrm{N}(U)} \frac{1}{\mathrm{N}(S')} \\
&\leq \left(\sum_{U\in\mathcal{A}_r(\frac{1}{2},v)} \frac{1}{\mathrm{N}(U)} \right) \left(\sum_{\substack{S' \\ x<\mathrm{N}(S')<xy}} \frac{1}{\mathrm{N}(S')} \right).
\end{aligned}$$

By using the method as in the proof of Lemma 2.2.4, we estimate the number of ideals whose norms are less than or equal to a given integer N. Then we have

$$\sum_{\substack{S' \\ x<\mathrm{N}(S')<xy}} \frac{1}{\mathrm{N}(S')} \ll \ln y \quad as\ y \to \infty.$$

From Lemma 2.2.7 we have

$$\sum_{U\in\mathcal{A}_r(\frac{1}{2},v)} \frac{1}{\mathrm{N}(U)} \ll \frac{\ln(1+g((r)))}{e^{v^{\beta(1-\varepsilon)}}} \quad as\ v \to \infty,$$

and then

$$\sum_{U\in\mathcal{A}_r(\frac{1}{2},v)} \sum_{\substack{(s)^v \\ (s,r)=U \\ x\mathrm{N}(U)<|s|^2<xy\mathrm{N}(U)}} \frac{1}{|s|^2} \ll \frac{\ln(1+g((r)))\ln y}{e^{v^{\beta(1-\varepsilon)}}} \quad as\ v\ \to \infty. \tag{2.14}$$

Thus we get the desired estimate for the first term of the right side of (2.13) with $U \in \mathcal{A}_r(1/2, v)$.

Now we consider the second term of the right side of (2.13) with $U \in \mathcal{B}_r(1/2, v)$ and $g((s)) = v$. In this case we have

$$1 \leqslant \sum_{\substack{P|(s) \\ \mathrm{N}(P)\geq v=g((s))}} \frac{1}{\mathrm{N}(P)} \leq \sum_{\substack{P|U \\ \mathrm{N}(P)\geq v}} \frac{1}{\mathrm{N}(P)} + \sum_{\substack{P|S' \\ \mathrm{N}(P)\geq v}} \frac{1}{\mathrm{N}(P)} < \frac{1}{2} + \sum_{\substack{P|S' \\ \mathrm{N}(P)\geq v}} \frac{1}{\mathrm{N}(P)},$$

which shows $\sum_{\substack{P|S' \\ \mathrm{N}(P)\geq v}} \mathrm{N}^{-1}(P) > 1/2$. From Lemma 2.2.6, we see

$$\sum_{\substack{S' \\ x<\mathrm{N}(S')\leqslant 2x}} \frac{1}{\mathrm{N}(S')} < \left(\sum_{\substack{S' \\ x<\mathrm{N}(S')\leqslant 2x}} 1 \right) \frac{1}{x} \leqslant \frac{\#\mathcal{N}(\frac{1}{2}, v, 2x)}{x} \ll \frac{1}{e^{v^{\beta(1-\varepsilon)}}} \quad as\ v \to \infty.$$

Thus we have

$$\begin{aligned}
\sum_{\substack{S' \\ x<\mathrm{N}(S')<xy}} \frac{1}{\mathrm{N}(S')} &\leq \sum_{k=1}^{[y]} \sum_{\substack{S' \\ kx<\mathrm{N}(S')\leq(k+1)x}} \frac{1}{\mathrm{N}(S')} \\
&< \frac{1}{x}\sum_{k=1}^{[y]} \frac{1}{k} \left(\sum_{\substack{S' \\ kx<\mathrm{N}(S')\leqslant(k+1)x}} 1 \right) \\
&\leqslant \frac{1}{x}\left(1-\frac{1}{2}\right)\left(\#\mathcal{N}\left(\frac{1}{2}, v, 2x\right)\right) + \frac{1}{x}\left(\frac{1}{2}-\frac{1}{3}\right)\left(\#\mathcal{N}\left(\frac{1}{2}, v, 3x\right)\right) \\
&\quad + \cdots + \frac{1}{x}\frac{1}{[y]}\left(\#\mathcal{N}\left(\frac{1}{2}, v, ([y]+1)x\right)\right) \\
&\ll \left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{[y]}\right)\cdot\frac{1}{e^{v^{\beta(1-\varepsilon)}}} \\
&\ll \ln y \cdot \frac{1}{e^{v^{\beta(1-\varepsilon)}}} \quad as\ y \to \infty.
\end{aligned}$$

This gives the estimate for the second terms as follows:

$$\begin{aligned}
\sum_{U\in\mathcal{B}_r(\frac{1}{2},v)} \sum_{\substack{(s)^{(v)}\\(s,r)=U\\ x\mathrm{N}(U)<|s|^2<xy\mathrm{N}(U)}} \frac{1}{|s|^2} &= \sum_{U\in\mathcal{B}_r(\frac{1}{2},v)} \sum_{\substack{(US')^{(v)}\\(s,r)=U\\ x<\mathrm{N}(S')<xy}} \frac{1}{\mathrm{N}(U)}\frac{1}{\mathrm{N}(S')} \\
&\leqslant \left(\sum_{U\in\mathcal{B}_r(\frac{1}{2},v)} \frac{1}{\mathrm{N}(U)}\right)\left(\sum_{\substack{S'\\ x<\mathrm{N}(S')<xy}} \frac{1}{\mathrm{N}(S')}\right) \\
&\ll \left(\sum_{U\in\mathcal{B}_r(\frac{1}{2},v)} \frac{1}{\mathrm{N}(U)}\right)\frac{\ln y}{e^{v^{\beta(1-\varepsilon)}}} \\
&\leqslant \left(\sum_{U|(r)} \frac{1}{\mathrm{N}(U)}\right)\frac{\ln y}{e^{v^{\beta(1-\varepsilon)}}} \\
&\leq \left(\prod_{P|(r)} \left(1-\frac{1}{\mathrm{N}(P)}\right)^{-1}\right)\frac{\ln y}{e^{v^{\beta(1-\varepsilon)}}} \\
&= \frac{|r|^2}{\Phi((r))}\frac{\ln y}{e^{v^{\beta(1-\varepsilon)}}} \\
&\ll \ln(1+g((r)))\frac{\ln y}{e^{v^{\beta(1-\varepsilon)}}} \quad as\ v\to\infty. \qquad (2.15)
\end{aligned}$$

Hence we can deduce the assertion of Lemma 2.2.8 from (2.14) and (2.15). □

2.3 Proof of main result

Now we will give the proof of Theorem 2.1.3.

Proof of Theorem 2.1.3. Let $r, s \in \mathbb{Z}[\omega]$ be two integers such that the two principal ideals (r) and (s) are different. Put
$\delta = \min\left\{\frac{\Psi((r))}{|r|}, \frac{\Psi((s))}{|s|}\right\}$, $\Delta = \max\left\{\frac{\Psi((r))}{|r|}, \frac{\Psi((s))}{|s|}\right\}$, and $t = \max\{g((r)), g((s))\}$.
Let $\mathcal{R}_a$ and $\mathcal{S}_b$ for $a, b \in \mathbb{Z}[\omega]$ be

$$\mathcal{R}_a = \left\{z : \left|z - \frac{a}{r}\right| < \frac{\Psi((r))}{|r|}, z \in \mathbb{F}\right\},\ \mathcal{S}_b = \left\{z : \left|z - \frac{b}{s}\right| < \frac{\Psi((s))}{|s|}, z \in \mathbb{F}\right\}$$

for given r and s. Then

$$\mathcal{E}_{(r)} = \bigcup_{\substack{a\in\mathbb{Z}[\omega]\\ \frac{a}{r}\in\mathbb{F}\\ (a,r)=(1)}} \mathcal{R}_a\ ,\ \mathcal{E}_{(s)} = \bigcup_{\substack{b\in\mathbb{Z}[\omega]\\ \frac{b}{s}\in\mathbb{F}\\ (b,s)=(1)}} \mathcal{S}_b.$$

If $\Psi((r)) \leqslant 1/2$ and $\Psi((s)) \leqslant 1/2$, then for any a_1 and a_2 with $a_1 \neq a_2$, we see $\mathcal{R}_{a_1} \cap \mathcal{R}_{a_2} = \emptyset$ and the same holds for $\mathcal{S}_b$. Then we have

$$\begin{aligned} \lambda(\mathcal{E}_{(r)} \cap \mathcal{E}_{(s)}) &= \sum_{\substack{\frac{a}{r}\in\mathbb{F} \\ (a,r)=(1)}} \sum_{\substack{\frac{b}{s}\in\mathbb{F} \\ (b,s)=(1)}} \lambda(\mathcal{R}_a \cap \mathcal{S}_b) \\ &\leq \delta^2 \sum_{\substack{\frac{a}{r}\in\mathbb{F} \\ (a,r)=(1)}} \sum_{\substack{\frac{b}{s}\in\mathbb{F} \\ (b,s)=(1) \\ |\frac{a}{r}-\frac{b}{s}|<\Delta}} 1 \\ &= \delta^2 \sum_{\substack{\frac{a}{r}\in\mathbb{F} \\ (a,r)=(1)}} \sum_{\substack{\frac{b}{s}\in\mathbb{F} \\ (b,s)=(1) \\ |as-br|<|r||s|\Delta}} 1. \end{aligned} \tag{2.16}$$

We define $H(k)$ as a set of pairs of integers $a, b \in \mathbb{Z}[\omega]$ by

$$H(k) = \left\{ \{a, b\} : as - br = k, (a, r) = (b, s) = (1),\ with\ \frac{a}{r}, \frac{b}{s} \in \mathbb{F} \right\}.$$

We denote by $\#H(k)$ the cardinality of $H(k)$ and we will estimate $\#H(k)$. Let $U = (r, s)$ and S' and R' be ideals determined by $(s) = US'$ and $(r) = UR'$. Since $((a), R') = (1)$ and $(S', R') = (1)$, we have $(a)S' \neq (b)R'$, which shows $\#H(0) = 0$. Since $U \mid (as)$ and $U \mid (br)$ imply $U \mid (k)$, we have $\#H(k) = 0$ if $U \nmid (k)$. So we only need to consider $k \in \mathbb{Z}[\omega]$ with $U \mid (k)$. In this case, the principal ideal (k) can be uniquely represented as $(k) = U \cdot U_{(k)} \cdot K_1$. Here $U_{(k)}$ is the ideal whose all prime ideal factors are also the prime ideal factors of U and $(K_1, U) = (1)$.

If $(K_1, UR'S') \neq (1)$, then we can find some prime ideal P such that $P \mid K_1$ and $P \mid UR'S'$. Since $(K_1, U) = (1)$, either $P \mid R'$ or $P \mid S'$ holds. If $P \mid R'$, we see $P \mid (br)$ and $P \nmid (s)$. Here $P \mid R'$ implies $P \nmid (a)$ and we have $P \nmid (as)$, which is impossible since $P \mid (k)$. We can use the same approach for the case of $P \mid S'$ and get the same conclusion. Hence if $(K_1, UR'S') \neq (1)$, then we have $\#H(k) = 0$.

If $(U_{(k)}, R'S') \neq (1)$, then we can find some prime ideal P with $P \mid U_{(k)}$ and $P \mid R'S'$. If $P \mid R'$, then there exists a positive integer n such that $P^n \mid U$ and $P^{n+1} \nmid U$. So we see $P^{n+1} \mid (r)$, which means $br \in P^{n+1}$. From $P^{n+1} \mid (k)$, we see $P \mid (a)$, which is impossible since $((a), R') = (1)$ and $P \mid R'$. We can use the same method for the case $P \mid S'$ and get the same conclusion. So if $(U_{(k)}, R'S') \neq (1)$, then $\#H(k) = 0$.

Consequently we only need to estimate $\#H(k)$ in the case of $(K_1, UR'S') = (1)$, $(U_{(k)}, R'S') = (1)$ and $\mathrm{N}(U) \leqslant |k|^2$. Suppose $\{a_1, b_1\}$ and $\{a_2, b_2\}$ are two different pairs of integers in $H(k)$ for a given $k \in \mathbb{Z}[\omega]$. Then $(a_1 - a_2)(s) = (b_1 - b_2)(r)$. So we have

$$R' \mid (a_1 - a_2) \ \text{ and } \ S' \mid (b_1 - b_2). \tag{2.17}$$

This means that a_1 and a_2 are in the same residue class modulo R' as well as b_1 and b_2 are in the same residue class modulo S'. We consider the set of pairs (a, b) with $a/r, b/s \in \mathbb{F}$ such that each two of them satisfy (2.17). In order to estimate the cardinality of this set, let us consider the number of a_i of the left side of (2.17). The ideal R' can be represented by the standard basis form that there exist $r_1, r_2 \in \mathbb{Z}(\sqrt{d})$ such that $R' = [r_1, r_2] = \{x \cdot r_1 + y \cdot r2 : x, y \in \mathbb{Z}\}$. Let $\mathbb{Z}(\sqrt{d}) = W = [1, \omega]$, then there exists a rational integer matrix

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

whose terms are all rational integers such that $R' = MW$. Also, for the matrix M there exists two rational integer matrices M_l, M_r with $\det(M_l) = \pm 1$ and $\det(M_r) = \pm 1$ such that

$$M_l M M_r = \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix}, \quad e_1, e_2 \in \mathbb{Z}.$$

Here $\det(\cdot)$ denotes the determinant of metrix. Let $\alpha = [\alpha_1, \alpha_2], \beta = [\beta_1, \beta_2]$ with $\alpha = M_l R'$ and $W = M_r \beta$. Then we see

$$\alpha = \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix} \beta.$$

Thus we can consider α and β instead of R' and W. Since for any $c \in \mathbb{Z}(\sqrt{d})$ there exist $x_1, x_2 \in \mathbb{Z}$ such that $c = x_1\beta_1 + x_2\beta_2$, if $a_1 \equiv a_2 \pmod{R'}$ then there are $x_1, x_2, x_1', x_2' \in \mathbb{Z}$ such that

$$a_1 = x_1\beta_1 + x_2\beta_2, \ a_2 = x_1'\beta_1 + x_2'\beta_2. \tag{2.18}$$

Since $a_1 - a_2 \in R'$, there exist $t_1, t_2 \in \mathbb{Z}$ such that

$$a_1 - a_2 = t_1\alpha_1 + t_2\alpha_2 = t_1 e_1 \beta_1 + t_2 e_2 \beta_2.$$

By (2.18), we have

$$x_1 \equiv x_1' \pmod{e_1}, \ x_2 \equiv x_2' \pmod{e_2}.$$

From [27], we see $\mathrm{N}(R') = e_1 e_2$ and it is the number of residue classes modulo R'. Next, we estimate the number of $a \in \mathbb{Z}[\omega]$ such that $a/r \in \mathbb{F}$. We express a/r under the basis

$[1/r, \omega/r]$ and for each $a \in \mathbb{Z}[\omega]$, there are $a', a'' \in \mathbb{Z}$ such that $a = a' \cdot \frac{1}{r} + a'' \cdot \frac{\omega}{r}$. Then this problem is equivalent to count the lattice points under the basis $[1/r, \omega/r]$ in the fundamental area, and we see that it is exactly $|r|^2$. If there exist two different lattice points $a_1/r, a_2/r \in \mathbb{F}$ with $(r) \mid (a_1 - a_2)$, then $\mathrm{N}(a_1 - a_2) > |r|^2$ which shows that either a_1/r or a_2/r are out of the fundamental area. So each $a \in \mathbb{Z}[\omega]$ with $a/r \in \mathbb{F}$ is in the different residue class modulo (r). Then we see that the cardinality of the set of pairs (a, b) with $a/r, b/s \in \mathbb{F}$ such that each two of them satisfy (2.17), is $|r|^2\mathrm{N}^{-1}(R') = \mathrm{N}(U)$.

Next, we estimate the number of pairs of integers a, b in the above set with $(a, U) = (1)$ and $(b, U) = (1)$. For this reason we consider the pairs of integers a, b with $(a, U) \neq (1)$ or $(b, U) \neq (1)$ and exclude them from the pairs of integers a, b in the above set with $|a| \leqslant |r|$ and $|b| \leqslant |s|$. Here we assume a_j, b_j and a_l, b_l are two different pairs of solutions of (2.17). Now we estimate the nunber of pairs of integers a, b with $(a, U) \neq (1)$ or $(b, U) \neq (1)$. Since U can be decomposed into $U = P_1^{\gamma_1} P_2^{\gamma_2} \cdots P_j^{\gamma_j}$, we consider two cases of $P (= P_j)$.

Case 1. $P \mid U$, $P \nmid U_{(k)}$, and $P \nmid R'S'$.

We will show that $P \mid (a_j)$ implies $P \nmid (b_j)$, which means that integers a, b are in the different residue class modulo P. Indeed, since $R' \mid (a_j - a_l)$, $S' \mid (b_j - b_l)$ and $\gcd(\mathrm{N}(P), \mathrm{N}(R')) = \gcd(\mathrm{N}(P), \mathrm{N}(S')) = 1$, we have $P \mid (a_j - a_l)$ and $P \mid (b_j - b_l)$. These show $UP \nmid (k)$ and $UP \mid (a_j s)$, which means $UP \nmid (b_j r)$ and thus $P \nmid (b_j)$.

Case 2. $P \mid U$ and either $P \mid U_{(k)}$ or $P \mid R'S'$.

(i) $P \mid U_{(k)}$ and $P \nmid R'S'$.

As the same discussions in case 1, we have $P \mid (a_j - a_l)$ and $P \mid (b_j - b_l)$. Since $UP \mid (k)$ and $UP \mid (a_j s)$, we have $UP \mid (b_j r)$, which implies $P \mid (b_j)$. So in this case, $P \mid (a_j)$ implies $P \mid (b_j)$, which means that integers a, b are in the same residue class modulo P.

(ii) $P \mid R'S'$.

Assume $P \mid R'$ and $P \nmid S'$. Note that $P \nmid (a_j)$ holds in this case. Since $(P, S') = (1)$, all the integers b will be in the same residue class modulo P. In this case, we only need to exclude the pairs of integers a, b with $P \mid (b)$. Similarly, for the case of $P \nmid R'$ and $P \mid S'$, we only need to exclude the pairs of integers a, b with $P \mid (a)$.

From the above discussion, we have

$$
\begin{aligned}
\#H(k) &\leq \mathrm{N}(U) \prod_{\substack{P|U \\ P\nmid U_{(k)} \\ P\nmid R'S'}} \left(1-\frac{2}{\mathrm{N}(P)}\right) \prod_{\substack{P|U \\ P|U_{(k)}R'S'}} \left(1-\frac{1}{\mathrm{N}(P)}\right) \\
&\leq \mathrm{N}(U) \prod_{\substack{P|U \\ P\nmid U_{(k)} \\ P\nmid R'S'}} \left(1-\frac{1}{\mathrm{N}(P)}\right) \prod_{\substack{P|U \\ P\nmid U_{(k)} \\ P\nmid R'S'}} \left(1-\frac{1}{\mathrm{N}(P)}\right) \prod_{\substack{P|U \\ P\nmid U_{(k)} \\ P|R'S'}} \left(1-\frac{1}{\mathrm{N}(P)}\right) \\
&\cdot \prod_{\substack{P|U \\ P|U_{(k)} \\ P\nmid R'S'}} \left(1-\frac{1}{\mathrm{N}(P)}\right) \prod_{\substack{P|U \\ P|U_{(k)} \\ P|R'S'}} \left(1-\frac{1}{\mathrm{N}(P)}\right) \\
&= \mathrm{N}(U) \prod_{\substack{P|U \\ P\nmid U_{(k)}}} \left(1-\frac{1}{\mathrm{N}(P)}\right) \prod_{\substack{P|U \\ P\nmid R'S'}} \left(1-\frac{1}{\mathrm{N}(P)}\right) \\
&= \Phi(U) \prod_{\substack{P|U \\ P\nmid R'S'}} \left(1-\frac{1}{\mathrm{N}(P)}\right) \prod_{P|U_{(k)}} \left(1-\frac{1}{\mathrm{N}(P)}\right)^{-1}. \qquad (2.19)
\end{aligned}
$$

Now we use some notations, following Vaaler's method, see [28]:

$$
\begin{aligned}
\mathfrak{J}_0 &= \{P : P \mid U, P \nmid R'S'\}, \\
\mathfrak{J}_1 &= \{P : P \in \mathfrak{J}_0, \mathrm{N}(P) \leqslant t\}, \\
\mathfrak{J}_2 &= \{P : P \in \mathfrak{J}_0, \mathrm{N}(P) > t\}, \\
\mathfrak{I}_m &= \{I : I = P_1^{\gamma_1} P_2^{\gamma_2} \cdots P_k^{\gamma_k}, P_1, P_2, ..., P_k \in \mathfrak{J}_m, \gamma_1, \gamma_2, ..., \gamma_k \in \mathbb{Z}\} \\
& \quad\ \textit{with } m = 0, 1, 2.
\end{aligned}
$$

Since $U_{(k)} \in \mathfrak{I}_0$, we divide $U_{(k)}$ into two parts of $I_1 \in \mathfrak{I}_1$ and $I_2 \in \mathfrak{I}_2$, with $U_{(k)} = I_1 I_2$. Then, together with (2.19), we have the following estimate:

$$
\begin{aligned}
\#H(k) &\leqslant \Phi(U) \prod_{P\in\mathfrak{J}_0} \left(1-\frac{1}{\mathrm{N}(P)}\right) \prod_{P|I_1 I_2} \left(1-\frac{1}{\mathrm{N}(P)}\right)^{-1} \\
&= \Phi(U) \prod_{P\in\mathfrak{J}_1} \left(1-\frac{1}{\mathrm{N}(P)}\right) \prod_{P|I_1} \left(1-\frac{1}{\mathrm{N}(P)}\right)^{-1} \frac{\prod_{P\in\mathfrak{J}_2} \left(1-\frac{1}{\mathrm{N}(P)}\right)}{\prod_{P|I_2} \left(1-\frac{1}{\mathrm{N}(P)}\right)} \\
&\leq \Phi(U) \prod_{P\in\mathfrak{J}_1} \left(1-\frac{1}{\mathrm{N}(P)}\right) \prod_{P|I_1} \left(1-\frac{1}{\mathrm{N}(P)}\right)^{-1}. \qquad (2.20)
\end{aligned}
$$

Let

$$K = I_2 K_1 \ , \ Q = \prod_{\substack{P | R'S'U \\ \mathrm{N}(P) \leqslant t}} P.$$

Since $(K_1, U) = (1)$ and $(U_{(k)}, R'S') = (1)$, we have $(K_1, R'S'U) = (1)$ and $(I_2, R'S') = (1)$, which imply $(K, Q) = (1)$. Then by using (2.16) and (2.20), we get

$$\begin{aligned}
\lambda(\mathcal{E}_{(r)} \cap \mathcal{E}_{(s)}) &\leq \delta^2 \sum_{\substack{k \in \mathbb{Z}[\omega] \\ 1 \leqslant |k| \leqslant |r||s|\Delta}} \# H(k) \\
&\leq \delta^2 \sum_{I_1 \in \mathfrak{I}_1} \sum_{\substack{K \\ 1 \leqslant \mathrm{N}(K) \leqslant \frac{|r|^2 |s|^2 \Delta^2}{\mathrm{N}(U)\mathrm{N}(I_1)} \\ (K,Q)=(1)}} \Phi(U) \prod_{P \in \mathfrak{J}_1} \left(1 - \frac{1}{\mathrm{N}(P)}\right) \\
&\quad \cdot \prod_{P | I_1} \left(1 - \frac{1}{\mathrm{N}(P)}\right)^{-1} \\
&= \delta^2 \Phi(U) \prod_{P \in \mathfrak{J}_1} \left(1 - \frac{1}{\mathrm{N}(P)}\right) \\
&\quad \cdot \sum_{I_1 \in \mathfrak{I}_1} \left(\prod_{P | I_1} \left(1 - \frac{1}{\mathrm{N}(P)}\right)^{-1} \sum_{\substack{K \\ 1 \leqslant \mathrm{N}(K) \leqslant \frac{|r|^2 |s|^2 \Delta^2}{\mathrm{N}(U)\mathrm{N}(I_1)} \\ (K,Q)=(1)}} 1 \right). \qquad (2.21)
\end{aligned}$$

By the Landau prime ideal theorem 2.2.3, we have $(\pi(y)(\ln 2 + \ln y) + \ln \ln y) y^{-1} \ll 1$ as $y \to \infty$. Then there exists $b \geq 0$ such that for any $y \geq b$, we have $\pi(y)(\ln 2 + \ln y) + \ln \ln y \leqslant y \ln 3$. We will estimate

$$\sum_{|r|^2 \in \mathbf{Z}} \sum_{\substack{|s|^2 \in \mathbf{Z} \\ (r) \neq (s)}} \lambda(\mathcal{E}_{(r)} \cap \mathcal{E}_{(s)})$$

by dividing it into two cases.

<u>Case A</u>. $t \geq b$ and $|r|^2 |s|^2 \Delta^2 \geq 3^t \mathrm{N}(U)$.

By the sieve method for the imaginary quadratic fields and binomial theorem, we see

$$\begin{aligned}
\sum_{\substack{K \\ 1\leqslant \mathrm{N}(K)\leqslant \frac{|r|^2|s|^2\Delta^2}{\mathrm{N}(U)\mathrm{N}(I_1)} \\ (K,Q)=(1)}} 1 &= \sum_{D|Q} \mu(D) T\left(\left[\frac{|r|^2|s|^2\Delta^2}{\mathrm{N}(U)\mathrm{N}(I_1)\mathrm{N}(D)}\right]\right) \\
&\ll \sum_{D|Q} \mu(D)\left[\frac{|r|^2|s|^2\Delta^2}{\mathrm{N}(U)\mathrm{N}(I_1)\mathrm{N}(D)}\right] \\
&= \sum_{D|Q} \frac{\mu(D)}{\mathrm{N}(D)} \frac{|r|^2|s|^2\Delta^2}{\mathrm{N}(U)\mathrm{N}(I_1)} - \sum_{D|Q} \mu(D)\left\{\frac{|r|^2|s|^2\Delta^2}{\mathrm{N}(U)\mathrm{N}(I_1)\mathrm{N}(D)}\right\} \\
&\leqslant \frac{|r|^2|s|^2\Delta^2}{\mathrm{N}(U)\mathrm{N}(I_1)} \prod_{P|Q}\left(1-\frac{1}{\mathrm{N}(P)}\right) + \sum_{D|Q} |\mu(D)| \\
&\leqslant \frac{|r|^2|s|^2\Delta^2}{\mathrm{N}(U)\mathrm{N}(I_1)} \prod_{P|Q}\left(1-\frac{1}{\mathrm{N}(P)}\right) + 2^{\pi(t)} \quad as\ t \to \infty, \qquad (2.22)
\end{aligned}$$

where μ is the ideal version of Möbius function, that is,

$$\mu(D) = \begin{cases} (-1)^k, & \text{if } D = P_1 P_2 \cdots P_k, \\ 0, & \text{if } \exists P \text{ such that } P^2 \mid D, \end{cases}$$

and $T(\cdot)$ is the function we have used in the proof of Lemma 2.2.4. Next we use the 3rd Mertens' theorem 2.2.2 for an algebraic number field, we have

$$\begin{aligned}
2^{\pi(t)} &\leq \frac{3^t}{t^{\pi(t)} \ln t} \leqslant \frac{|r|^2|s|^2\Delta^2}{\mathrm{N}(U)} \frac{1}{t^{\pi(t)} \ln t} \\
&\ll \frac{|r|^2|s|^2\Delta^2}{\mathrm{N}(U)} \prod_{P|Q}\left(1-\frac{1}{\mathrm{N}(P)}\right) \frac{1}{t^{\pi(t)}} \quad as\ t \to \infty. \qquad (2.23)
\end{aligned}$$

If $\mathrm{N}(I_1) \leqslant t^{\pi(t)}$, we take (2.22) and (2.23) into (2.21) and get

$$\begin{aligned}
\lambda(\mathcal{E}_{(r)} \cap \mathcal{E}_{(s)}) &\ll \delta^2 \frac{\Phi(U)}{\mathrm{N}(U)} |r|^2 |s|^2 \Delta^2 \prod_{P|Q} \left(1 - \frac{1}{\mathrm{N}(P)}\right) \\
&\cdot \prod_{P \in \mathfrak{J}_1} \left(1 - \frac{1}{\mathrm{N}(P)}\right) \sum_{I_1 \in \mathfrak{I}_1} \frac{1}{\mathrm{N}(I_1) \prod_{P|I_1} \left(1 - \frac{1}{\mathrm{N}(P)}\right)} \\
&= \delta^2 \frac{\Phi(U)}{\mathrm{N}(U)} |r|^2 |s|^2 \Delta^2 \prod_{P|Q} \left(1 - \frac{1}{\mathrm{N}(P)}\right) \prod_{P \in \mathfrak{J}_1} \left(1 - \frac{1}{\mathrm{N}(P)}\right) \sum_{I_1 \in \mathfrak{I}_1} \frac{1}{\Phi(I_1)} \\
&\ll \delta^2 \frac{\Phi(U)}{\mathrm{N}(U)} |r|^2 |s|^2 \Delta^2 \prod_{P|Q} \left(1 - \frac{1}{\mathrm{N}(P)}\right) \\
&\cdot \prod_{P \in \mathfrak{J}_1} \left(1 - \frac{1}{\mathrm{N}(P)}\right) \left(1 + \frac{\mathrm{N}(P)}{(\mathrm{N}(P)-1)^2}\right) \\
&\leqslant \frac{\Psi^2(r)\Psi^2(s)\Phi(U)}{\mathrm{N}(U)} \prod_{P|R'S'U} \left(1 - \frac{1}{\mathrm{N}(P)}\right) \\
&\cdot \prod_{P \in \mathfrak{J}_1} \left(1 + \frac{1}{\mathrm{N}(P)(\mathrm{N}(P)-1)}\right) \\
&\leqslant \Phi((r)) \frac{\Psi^2((r))}{|r|^2} \Phi((s)) \frac{\Psi^2((s))}{|s|^2} \prod_{\substack{P|U \\ P \nmid R'S' \\ \mathrm{N}(P) \leqslant t}} \left(1 + \frac{1}{\mathrm{N}(P)(\mathrm{N}(P)-1)}\right) \\
&\ll \lambda\left(\mathcal{E}_{(r)}\right) \lambda\left(\mathcal{E}_{(s)}\right) \quad \mathit{as}\ t \to \infty. \qquad (2.24)
\end{aligned}$$

If $\mathrm{N}(I_1) > t^{\pi(t)}$, then there exist a prime ideal $P \in \mathfrak{J}_1$ and $\gamma \in \mathbb{Z}$ such that $P^\gamma \mid I_1$, $\mathrm{N}(P) \leqslant t$, and $(\mathrm{N}(P))^\gamma > t$. This implies that there exists an ideal D such that $D^2 \mid I_1$

and $(\mathrm{N}(D))^2 \geq t^{\frac{2}{3}}$. Then we have

$$\begin{aligned}
\lambda(\mathcal{E}_{(r)} \cap \mathcal{E}_{(s)}) &\ll \delta^2 \Phi(U) \prod_{P \in \mathfrak{J}_1} \left(1 - \frac{1}{\mathrm{N}(P)}\right) \sum_{\substack{I_1 \in \mathfrak{J}_1 \\ \mathrm{N}(I_1) > t^{\pi(t)}}} \frac{\mathrm{N}(I_1)}{\Phi(I_1)} \\
&\quad \cdot \sum_{\substack{K \\ 1 \leq \mathrm{N}(K) \leq \frac{|r|^2|s|^2\Delta^2}{\mathrm{N}(U)\mathrm{N}(I_1)} \\ (K,Q)=(1)}} 1 \\
&\ll \delta^2 \Phi(U) \sum_{\substack{I_1 \in \mathfrak{J}_1 \\ \mathrm{N}(I_1) > t^{\pi(t)}}} \sum_{\substack{K \\ 1 \leqslant \mathrm{N}(K) \leqslant \frac{|r|^2|s|^2\Delta^2}{\mathrm{N}(U)\mathrm{N}(I_1)}}} 1 \\
&\leq \delta^2 \Phi(U) \sum_{\substack{D \\ [t^{\frac{1}{3}}] \leqslant \mathrm{N}(D) < \infty}} \sum_{\substack{J \\ D^2 | J \\ 1 \leqslant \mathrm{N}(J) \leqslant \frac{|r|^2|s|^2\Delta^2}{\mathrm{N}(U)}}} 1 \\
&\ll \delta^2 \Delta^2 \frac{\Phi(U)}{\mathrm{N}(U)} |r|^2|s|^2 \sum_{\substack{D \\ [t^{\frac{1}{3}}] \leqslant \mathrm{N}(D) < \infty}} \frac{1}{(\mathrm{N}(D))^2} \quad as\ t \to \infty. \qquad (2.25)
\end{aligned}$$

We use a method similar to the proof of Lemma 2.2.4 to estimate the term $\sum_{[t^{\frac{1}{3}}] \leqslant \mathrm{N}(D) < \infty} \mathrm{N}^{-2}(D)$:

$$\sum_{[t^{\frac{1}{3}}] \leqslant \mathrm{N}(D) < \infty} \frac{1}{(\mathrm{N}(D))^2} \ll \sum_{n=[t^{\frac{1}{3}}]}^{\infty} \frac{1}{n^2} \ll \frac{1}{t^{\frac{1}{3}}} \quad as\ t \to \infty.$$

We take this estimate into (2.25) with Corollary 2.2.5 and get

$$\begin{aligned}
\lambda(\mathcal{E}_{(r)} \cap \mathcal{E}_{(s)}) &\ll \Psi^2((r))\Psi^2((s)) \frac{1}{t^{\frac{1}{3}}} \\
&\ll \Phi((r)) \frac{\Psi^2((r))}{|r|^2} (\ln t) \Phi((s)) \frac{\Psi^2((s))}{|s|^2} (\ln t) \frac{1}{t^{\frac{1}{3}}} \\
&\ll \lambda\left(\mathcal{E}_{(r)}\right) \lambda\left(\mathcal{E}_{(s)}\right) \frac{\ln^2 t}{t^{\frac{1}{3}}} \\
&\ll \lambda\left(\mathcal{E}_{(r)}\right) \lambda\left(\mathcal{E}_{(s)}\right) \quad as\ t \to \infty. \qquad (2.26)
\end{aligned}$$

Together with (2.24) and (2.26), we conclude, in Case A, that,

$$\sum_{\substack{|r|^2 \in \mathbf{Z} \\ (r) \neq (s)}} \sum_{|s|^2 \in \mathbf{Z}} \lambda(\mathcal{E}_{(r)} \cap \mathcal{E}_{(s)}) \ll \sum_{\substack{|r|^2 \in \mathbf{Z} \\ (r) \neq (s)}} \sum_{|s|^2 \in \mathbf{Z}} \lambda\left(\mathcal{E}_{(r)}\right) \lambda\left(\mathcal{E}_{(s)}\right). \qquad (2.27)$$

Case B. If $t < b$ or $|r|^2|s|^2\Delta^2 < 3^t \mathrm{N}(U)$.

Let $\eta_0 = e^{-\max\{b,C,v_0\}}$ and suppose $0 < \Lambda(\mathbf{Z}) \leqslant \eta_0$. We put $L = \ln(\frac{1}{\Lambda(\mathbf{Z})})$ and get

$$\begin{aligned}
\lambda(\mathcal{E}_{(r)} \cap \mathcal{E}_{(s)}) &\ll \Psi^2((r))\Psi^2((s))\frac{\Phi(U)}{\mathrm{N}(U)} \prod_{P\in\mathfrak{J}_1}\left(1-\frac{1}{\mathrm{N}(P)}\right)\sum_{I_1\in\mathfrak{I}_1}\frac{1}{\Phi(I_1)} \\
&< \Psi^2((r))\Psi^2((s)) \prod_{\substack{P|U \\ P\notin\mathfrak{J}_1}}\left(1-\frac{1}{\mathrm{N}(P)}\right) \\
&\cdot \prod_{P\in\mathfrak{J}_1}\left(1-\frac{1}{\mathrm{N}(P)}+\frac{1}{\mathrm{N}(P)(\mathrm{N}(P)-1)}-\frac{1}{(\mathrm{N}(P))^2(\mathrm{N}(P)-1)}\right) \\
&< \Psi^2((r))\Psi^2((s)) \\
&\ll \frac{\Psi^2((r))}{|r|^2}\Phi((r))\ln(1+g((r)))\frac{\Psi^2((s))}{|s|^2}\Phi((s))\ln(1+g((s))) \\
&\ll \lambda(\mathcal{E}_{(r)})\lambda(\mathcal{E}_{(s)})\ln^2(1+t) \quad as\ t\to\infty. \qquad (2.28)
\end{aligned}$$

If $t < L$, which implies $L \geq b$, then from (2.28) we have

$$\begin{aligned}
\sum_{\substack{|r|^2\in\mathbf{Z} \\ (r)\neq(s)}}\sum_{|s|^2\in\mathbf{Z}} \lambda\left(\mathcal{E}_{(r)}\cap\mathcal{E}_{(s)}\right) &\ll \sum_{\substack{|r|^2\in\mathbf{Z} \\ (r)\neq(s) \\ t<L}}\sum_{|s|^2\in\mathbf{Z}} \lambda\left(\mathcal{E}_{(r)}\right)\lambda\left(\mathcal{E}_{(s)}\right)\ln^2(1+t) \\
&< (\Lambda(\mathbf{Z}))^2\left(\ln\left(1+\ln\frac{1}{\Lambda(\mathbf{Z})}\right)\right)^2 \\
&\ll (\Lambda(\mathbf{Z}))^2\left(\ln\ln\frac{1}{\Lambda(\mathbf{Z})}\right)^2 \quad as\ t\to\infty. \qquad (2.29)
\end{aligned}$$

If $t \geq L$ and $\mathrm{N}(U) < |r|^2|s|^2\Delta^2 < 3^t\mathrm{N}(U)$, then

$$
\begin{aligned}
& \sum_{\substack{|r|^2\in\mathbf{Z} \\ (r)\neq(s)}} \sum_{|s|^2\in\mathbf{Z}} \lambda\left(\mathcal{E}_{(r)} \cap \mathcal{E}_{(s)}\right) \\
\ll\ & \sum_{\substack{|r|^2\in\mathbf{Z} \\ (r)\neq(s)}} \sum_{|s|^2\in\mathbf{Z}} \Psi^2((r))\Psi^2((s)) \\
\leqslant\ & \sum_{m=L}^{\infty} \sum_{n=1}^{m} \left(\sum_{\substack{(r)^m \\ |r|^2\in\mathbf{Z} \\ \mathrm{N}(U)<|r|^2|s|^2\Delta^2<3^m\mathrm{N}(U)}} \sum_{\substack{(s)^n \\ |s|^2\in\mathbf{Z}}} \Psi^2((r))\Psi^2((s)) \right) \\
=\ & \sum_{m=L}^{\infty} \sum_{n=1}^{m} \left(\sum_{\substack{(s)^n \\ |s|^2\in\mathbf{Z}}} \Psi^2((s)) \sum_{\substack{(r)^m \\ |r|^2\in\mathbf{Z} \\ \mathrm{N}(U)<|r|^2|s|^2\Delta^2<3^m\mathrm{N}(U)}} \Psi^2((r)) \right) \\
\ll\ & \sum_{m=L}^{\infty} \sum_{n=1}^{m} \left(\sum_{\substack{(s)^n \\ |s|^2\in\mathbf{Z}}} \lambda(\mathcal{E}_{(s)}) \ln(1+n) \sum_{\substack{(r)^m \\ \mathrm{N}(U)<|r|^2|s|^2\Delta^2<3^m\mathrm{N}(U)}} \Psi^2((r)) \right) \\
\ll\ & \sum_{m=L}^{\infty} \ln(1+m) \sum_{n=1}^{m} \left(\sum_{\substack{(s)^n \\ |s|^2\in\mathbf{Z}}} \lambda(\mathcal{E}_{(s)}) \sum_{\substack{(r)^m \\ \mathrm{N}(U)<|r|^2|s|^2\Delta^2<3^m\mathrm{N}(U)}} \Psi^2((r)) \right).
\end{aligned}
\tag{2.30}
$$

If $\Psi((r))|r|^{-1} \leqslant \Psi((s))|s|^{-1}$, then $\Delta = \Psi((s))|s|^{-1}$ and $|r|^2|s|^2\Delta^2 = |r|^2\Psi^2((s))$. By using Lemma 2.2.7 with $\xi = 1/2$ and $e^{1/2}(1-\varepsilon) = 3/2$, we have

$$
\begin{aligned}
\sum_{\substack{(r)^m \\ \mathrm{N}(U)<|r|^2\Psi^2((s))<3^m\mathrm{N}(U)}} \Psi^2((r)) &\ll C \sum_{\substack{(r)^m \\ \mathrm{N}(U)<|r|^2\Psi^2((s))<3^m\mathrm{N}(U)}} \frac{1}{|r|^2} \\
&\ll C(\ln(1+n))(\ln 3^m)e^{-m^{\beta(1-\varepsilon)}} \\
&\ll Cm(\ln(1+m))e^{-m^{\frac{3}{2}}},
\end{aligned}
\tag{2.31}
$$

where $C > 0$ is a constant which satisfies $\Psi((r)) \leqslant C|r|^{-1}$ for all principal ideals (r). This constant exists by the assumption $\Psi((r)) = \mathcal{O}(|r|^{-1})$. If $\Psi((r))|r|^{-1} > \Psi((s))|s|^{-1}$, then we can use the same approach as Vaaler's to divide the set $\mathbf{Z}$ into some small pieces, that is, let

$$W_j = \left\{ e : e \in \mathbb{Z}[\omega],\ \frac{C}{2^{j+1}} < |e|^2\Psi^2((e)) \leqslant \frac{C}{2^j} \right\}$$

with $j = 0, 1, 2....$ For $r \in W_j$ and $\mathrm{N}(U) < |s|^2\Psi^2((r)) < 3^m\mathrm{N}(U)$, we see

$$C|s|^2 2^{-j-1} 3^{-m}\mathrm{N}^{-1}(U) < |r|^2 < C|s|^2 2^{-j}\mathrm{N}^{-1}(U).$$

From Lemma 2.2.8, we have

$$\begin{aligned}
\sum_{\substack{(r)^m \\ \mathrm{N}(U)<|r|^2\Psi^2((s))<3^m\mathrm{N}(U)}} \Psi^2((r)) &\leq C\sum_{j=0}^{\infty}\frac{1}{2^j}\sum_{\substack{(r)^m \\ \frac{C|s|^2}{2^{j+1}3^m}\frac{1}{\mathrm{N}(U)}<|r|^2<\frac{C|s|^2}{2^j}\frac{1}{\mathrm{N}(U)}}}\frac{1}{|r|^2} \\
&\ll C\sum_{j=0}^{\infty}\frac{1}{2^j}\ln(1+g((s)))\ln(3^m)e^{-v^{\frac{3}{2}}} \\
&\ll Cm(\ln(1+m))e^{-m^{\frac{3}{2}}}. \qquad (2.32)
\end{aligned}$$

By using (2.30), (2.31) and (2.32), we find

$$\begin{aligned}
&\sum_{\substack{|r|^2\in\mathbf{Z} \\ (r)\neq(s)}}\sum_{|s|^2\in\mathbf{Z}}\lambda\left(\mathcal{E}_{(r)}\cap\mathcal{E}_{(s)}\right) \\
\ll\ & \sum_{m=L}^{\infty}\ln(1+m)\sum_{n=1}^{m}\left(\sum_{\substack{(s)^n \\ |s|^2\in\mathbf{Z}}}\lambda(\mathcal{E}_{(s)})Cm(\ln(1+m))e^{-m^{\frac{3}{2}}}\right) \\
=\ & C\sum_{m=L}^{\infty}m\ln^2(1+m)e^{-m^{\frac{3}{2}}}\left(\sum_{n=1}^{m}\sum_{\substack{(s)^n \\ |s|^2\in\mathbf{Z}}}\lambda(\mathcal{E}_{(s)})\right) \\
<\ & C\sum_{m=L}^{\infty}m\ln^2(1+m)e^{-m^{\frac{3}{2}}}\left(\sum_{n=1}^{\infty}\sum_{\substack{(s)^n \\ |s|^2\in\mathbf{Z}}}\lambda(\mathcal{E}_{(s)})\right) \\
\ll\ & \frac{1}{e^L}\Lambda(\mathbf{Z}) = (\Lambda(\mathbf{Z}))^2. \qquad (2.33)
\end{aligned}$$

Then (2.29) and (2.33) imply the following

$$\sum_{\substack{|r|^2\in\mathbf{Z} \\ (r)\neq(s)}}\sum_{|s|^2\in\mathbf{Z}}\lambda\left(\mathcal{E}_{(r)}\cap\mathcal{E}_{(s)}\right) \ll (\Lambda(\mathbf{Z}))^2\left(\ln\ln\frac{1}{\Lambda(\mathbf{Z})}\right)^2 \qquad (2.34)$$

in Case B. From (2.27) and (2.34), we get the assertion of Theorem 2.1.3. ☐

Now we will show Theorem 2.1.2.

Proof of Theorem 2.1.2. Since $\sum \Phi((r))\Psi^2((r))|r|^{-2} = \infty$, by using the Gallagher type result over an imaginary quadratic field, see [21], we have $\lim_{N\to\infty} \lambda(\bigcup_{|r|^2=N}^{\infty} \mathcal{E}_{(r)}) = 0$ or 1.

Suppose $\lim_{N\to\infty} \lambda(\bigcup_{|r|^2=N}^{\infty} \mathcal{E}_{(r)}) = 0$. This also implies

$$\lim_{|r|^2\to\infty} \lambda(\mathcal{E}_{(r)}) = 0. \tag{2.35}$$

We can choose a large rational integer m where $\lambda(\bigcup_{|r|^2=m}^{\infty} \mathcal{E}_{(r)}) \leqslant \frac{1}{4}\eta$. Let $j = \max\{k, m\}$. From $\sum \Phi((r))\Psi^2((r))|r|^{-2} = \sum_{|r|^2=1}^{\infty} \lambda(\mathcal{E}_{(r)}) = \infty$ and (2.35), it follows that there exists a finite subset $\mathbf{Z}$ of $\{j, j+1, j+2, ...\}$ such that $2/3\eta \leqslant \Lambda(\mathbf{Z}) \leqslant \eta$. Since $\bigcup_{|r|^2\in\mathbf{Z}} \mathcal{E}_{(r)} \subseteq \bigcup_{|r|^2=m} \mathcal{E}_{(r)}$, we have

$$\begin{aligned} \frac{1}{4}\eta &\geq \lambda(\bigcup_{|r|^2=m} \mathcal{E}_{(r)}) \geq \lambda(\bigcup_{|r|^2\in\mathbf{Z}} \mathcal{E}_{(r)}) \\ &\geq \sum_{|r|^2\in\mathbf{Z}} \lambda(\mathcal{E}_{(r)}) - \frac{1}{2} \sum_{\substack{|r|^2\in\mathbf{Z} \\ (r)\neq(s)}} \sum_{|s|^2\in\mathbf{Z}} \lambda\left(\mathcal{E}_{(r)} \cap \mathcal{E}_{(s)}\right) \\ &\geq \Lambda(\mathbf{Z}) - \frac{1}{2}\Lambda(\mathbf{Z}) \\ &\geq \frac{1}{3}\eta, \end{aligned}$$

which is impossible. This implies $\lim_{N\to\infty} \lambda(\bigcup_{|r|^2=N}^{\infty} \mathcal{E}_{(r)}) \neq 0$ which shows the assertion of Theorem 2.1.1. ☐

2.4 An example

In this section, we give an example following the example of [7], and show that the divergence condition in the Duffin-Schaeffer conjecture over imaginary quadratic fields is reasonable.

Let

$$\Sigma = \left\{(a, r) : a, r \in \mathbb{Z}[\omega], r \neq 0, \frac{a}{r} \in \mathbb{F}\right\}.$$

Define the sets

$$D_1 = \left\{z \in \mathbb{F} : \left|z - \frac{a}{r}\right| < \frac{\Psi((r))}{|r|} \; has\ infinitely\ many\ (a, r) \in \Sigma \; with\ (a, r) = (1)\right\}$$

and

$$D_2 = \left\{ z \in \mathbb{F} : \left| z - \frac{a}{r} \right| < \frac{\Psi((r))}{|r|} \text{ has infinitely many } (a, r) \in \Sigma \right\}.$$

We denote by λ in this section the normalized Lebesgue measure of $\mathbb{F}$, i.e., $\lambda(\mathbb{F}) = 1$, and give a sequence $\{\Psi((r))\}$ with $\sum_{r \in \mathbb{Z}[\omega] \setminus \{0\}} \Psi^2((r)) = \infty$ such that $\lambda(D_2) < 1$. First, we give the complex version of Lemma V in [7] as follows:

Lemma 2.4.1. *Let R and ε be given positive numbers. There is an infinite sequence $\{\Psi((r))\}$ of non-negative numbers with $\Psi((r)) = 0$ for all but finitely many r such that*

$$\sum \Psi^2((r)) > 1, \quad \sum \Phi((r)) \frac{\Psi^2((r))}{|r|^2} < c_d \varepsilon, \quad \Psi((r)) = 0 \text{ whenever } |r| \leqslant R,$$

where c_d is some constant depending on d, but for $z \in \mathbb{F}$ the inequality

$$\left| z - \frac{a}{r} \right| < \frac{\Psi((r))}{|r|}$$

for some $a, r \in \mathbb{Z}[\omega]$ can be satisfied only in a set of λ-measure smaller than ε.

Proof. Let N_d be the number of units of the imaginary quadratic field $\mathbb{Q}(\sqrt{d})$. Fix some $\alpha > 0$ with $\alpha < \sqrt{-d}\varepsilon/(2N_d k'(d)\pi)$ and we can choose prime numbers $p_1, p_2, \cdots, p_k$ such that

$$\prod_{i=1}^{k} \left(1 + \frac{1}{p_i}\right) > 1 + \frac{1}{N_d \alpha},$$

where $p_i > R$ for $1 \leqslant i \leqslant k$, since

$$\sum_{p:\text{prime}} \frac{1}{p}$$

diverges. Denote by (u) a principal ideal as

$$(u) = (p_1)(p_2) \cdots (p_k) = \prod_{P|(u)} P^c,$$

where P denotes the prime ideal and $c \geqslant 0$. Note that we do not need (p_i) be all prime ideals, and for any ideal U with $U \mid (u)$ it can be represented as $U = \prod_{P|U} P^{c'}$ with $0 \leqslant c' \leqslant c$. We define $\Psi((r))$ as follows:

$$\Psi((r)) = \begin{cases} \frac{\alpha^{1/2}|r|^{1/2}}{|u|^{1/2}}, & \text{if } |r| > 1 \text{ and } (r) \mid (u) \\ 0, & \text{otherwise.} \end{cases}$$

Define the set

$$E_{(r)} = \bigcup_{\substack{|a|^2 \leqslant |r|^2 \\ a \in \mathbb{Z}[\omega]}} \left\{ z \in \mathbb{F} : \left| z - \frac{a}{r} \right| < \frac{\Psi((r))}{|r|} \right\}$$

and put

$$E = \bigcup_{\substack{(r)|(u) \\ (r) \neq (1)}} E_{(r)}.$$

Since $E_{(r)} \subset E_{(u)}$ for all (r) with $(r)|(u)$, we have

$$\lambda(E) = \lambda(E_{(u)}) \leqslant \pi \frac{N_d \alpha}{|u|^2} \cdot \frac{2}{\sqrt{-d}} \cdot k'(d)|u|^2 = \frac{2N_d k'(d)\pi}{\sqrt{-d}} \alpha < \varepsilon.$$

Also we have

$$\begin{aligned}
\sum_{\substack{r \in \mathbb{Z}[\omega] \setminus \{0\} \\ (r)|(u) \\ (r) \neq (1)}} \Psi^2((r)) = \frac{\alpha}{|u|} \sum_{\substack{r \in \mathbb{Z}[\omega] \setminus \{0\} \\ (r)|(u) \\ (r) \neq (1)}} |r| \quad &\geqslant \quad N_d \frac{\alpha}{|u|} \left(\prod_{i=1}^{k} (1 + p_i) - 1 \right) \\
&\geqslant \quad N_d \alpha \left(\prod_{i=1}^{k} \left(1 + \frac{1}{p_i} \right) - 1 \right) > 1
\end{aligned}$$

and

$$\begin{aligned}
\sum_{\substack{r \in \mathbb{Z}[\omega] \setminus \{0\} \\ (r)|(u) \\ (r) \neq (1)}} \Phi((r)) \frac{\Psi^2((r))}{|r|^2} \quad &= \quad \frac{N_d \alpha}{|u|} \sum_{\substack{(r)|(u) \\ (r) \neq (1)}} \frac{\Phi((r))}{|r|} \leqslant \frac{N_d \alpha}{|u|} \sum_{\substack{U:ideals \\ U|(u)}} \frac{\Phi(U)}{(\mathrm{N}(U))^{1/2}} \\
&= \quad \frac{N_d \alpha}{|u|} \sum_{\substack{U:ideals \\ U|(u)}} \frac{1}{(\mathrm{N}(U))^{1/2}} \prod_{P|U} \Phi(P^{c'}) \\
&= \quad \frac{N_d \alpha}{|u|} \prod_{P|(u)} \left(1 + \frac{\Phi(P)}{(\mathrm{N}(P))^{1/2}} + \cdots + \frac{\Phi(P^c)}{(\mathrm{N}(P^c))^{1/2}} \right) \\
&< \quad \frac{2N_d \alpha}{|u|} \prod_{P|(u)} (\mathrm{N}(P))^{c/2} = 2N_d \alpha < \frac{\sqrt{-d}}{k'(d)\pi} \varepsilon.
\end{aligned}$$

Thus, we see that the sequence $\{\Psi((r))\}$ with $\Psi((r))$ defined above is the required finite sequence. $\square$

Now let $R_1 = 1$ and we have a sequence $\{\Psi^{(1)}((r))\}$ which satisfies Lemma 2.4.1 with $R = R_1$ and $\varepsilon = 2^{-1}$. Then for some R_2 with $\Psi^{(1)}((r)) = 0$ for all $|r| \geqslant R_2$, let $R = R_2$ and $\varepsilon = 2^{-2}$ and we have another sequence $\{\Psi^{(2)}((r))\}$ which satisfies Lemma

2.4.1. We do this process infinitely many times and obtain infinitely many sequences of $\{\Psi^{(1)}((r))\}, \{\Psi^{(2)}((r))\}, \cdots, \{\Psi^{(n)}((r))\}, \cdots$. Let $\Psi((r)) = \sum_{k=1}^{\infty} \Psi^{(k)}((r))$ for all $r \in \mathbb{Z}[\omega]\backslash\{0\}$. Then we see

$$\sum_{r\in\mathbb{Z}[\omega]\backslash\{0\}} \Psi^2((r)) = \infty,$$

whereas

$$\sum_{r\in\mathbb{Z}[\omega]\backslash\{0\}} \Phi((r))\frac{\Psi^2((r))}{|r|^2} < \infty.$$

However, λ-measure of the set of $z \in \mathbb{F}$ satisfying inequality $|z - a/r| < \Psi((r))/|r|$ is smaller than 1 by our choice of $\{\Psi((r))\}$, which means $\lambda(D_2) < 1$. Thus even $\sum_{r\in\mathbb{Z}[\omega]\backslash\{0\}} \Psi^2((r)) = \infty$, we cannot ignore the possibility of the case $\lambda(D_2) = 0$, and from our choice of $\{\Psi((r))\}$ we see $\sum \Phi((r))\Psi^2((r))|r|^{-2} < \infty$ in this case.

Chapter 3

Hausdorff dimension of the exceptional set for complex numbers

3.1 Generalized Jarník and Besicovitch's theorem over imaginary quadratic fields

First, we define closed disc $\mathcal{I}$ in the complex plane by

$$\mathcal{I} = \left\{ z \in \mathbb{C} : \left| z - \frac{a}{r} \right| \leqslant \delta \right\},$$

where δ is a positive real number and $a, r \in \mathbb{Z}[\omega]$ with $r \neq 0$. We adopt the following as the definition of the Hausdorff dimension of a subset of complex numbers (see [3] and G. Harman [11] chapter 10).

Definition 2. *Suppose that D is a set of complex numbers. The Hausdorff dimension of D is equal to d ($\dim_H D = d$) if it satisfies the next two conditions:*
(i) For any $\beta > d$ and any $\varepsilon > 0$, there exists a sequence of closed discs $\{\mathcal{I}_j\}_{j=1}^{\infty}$ such that
(a) $D \subset \cup_{j=1}^{\infty} \mathcal{I}_j$,
(b) $\sum_{j=1}^{\infty} (\mathrm{diam}(\mathcal{I}_j))^{\beta} < 1$, where diam$(\cdot)$ denotes the diameter of the closed disc,
(c) $\mathrm{diam}(\mathcal{I}_j) < \varepsilon$, for any $j \in \mathbb{N}$.
(ii) For any $\beta < d$, there exists $\varepsilon > 0$ such that there is no sequence of closed discs satiesfying all of the above (a), (b) and (c).

Our main result is the following, which is a complex number version of Theorem 10.6 in [11].

Theorem 3.1.1. *For an infinite subset $\mathcal{A}$ of $\mathbb{Z}[\omega]\backslash\{0\}$, let*

$$\nu = \sup\left\{h \geqslant 0 : \sum_{r\in\mathcal{A}}\left(\frac{1}{|r|^2}\right)^h = \infty\right\}.$$

For a real number ρ with $\rho > \nu$, define the set

$$D = \left\{z \in \mathbb{F} : \left|z - \frac{a}{r}\right| < |r|^{-(1+\rho)} \text{ has infinitely many } (a, r) \in \Sigma \text{ with } r \in \mathcal{A}\right\}.$$

Then we have $\dim_H D = \frac{2(1+\nu)}{1+\rho}$.

If the class number of $\mathbb{Q}(\sqrt{d})$ is 1 and $\mathcal{A} = \mathbb{Z}[\omega]\backslash\{0\}$, then we have $\nu = 1$ and we see that for any $z \in D$ there exist infinitely many pairs of a and r in $\mathbb{Z}[\omega]$ with $r \neq 0$ such that $|z - a/r| < |r|^{-(1+\rho)}$ holds and $(a, r) = (1)$, where $(a, r) = (1)$ means that the ideals (a) and (r) are coprime. This is because of the following: (i) if $a'/r' = a/r$, $|z - a'/r'| < |r'|^{-(1+\rho)}$ and $|r'| > |r|$ hold, then $|z - a/r| < |r|^{-(1+\rho)}$ also holds, (ii) there are at most finitely many pairs of a' and r' with $a'/r' = a/r$ such that $|z - a'/r'| < |r'|^{-(1+\rho)}$ holds. Thus, in this case, there is no difference between the inequality with and without the coprime condition on a and r. This situation is the same as V. Jarnik and A. S. Besicovitch's result for real numbers. However, it seems to be not obvious if the class number is not 1.

Corollary 3.1.2. *Suppose that the class number of $\mathbb{Q}(\sqrt{d})$ is 1 and put*

$$D_0 = \left\{z \in \mathbb{F} : \left|z - \frac{a}{r}\right| < |r|^{-(1+\rho)} \text{ has infinitely many } (a, r) \in \Sigma \text{ with } (a, r) = (1)\right\}.$$

then $\dim_H D_0 = \frac{4}{1+\rho}$ *for* $\rho > 1$.

3.2 Generalized Harman's result over imaginary quadratic fields

We also consider the set of solutions related to the Duffin-Schaeffer conjecture for complex numbers from Theorem 3.1.1.

Theorem 3.2.1. *Suppose that* $\Psi((r))$ *is a non-negative function such that*

$$\sum_{r\in\mathbb{Z}[\omega]\backslash\{0\}} \Phi((r))\frac{\Psi^2((r))}{|r|^2}$$

diverges. Then we have $\dim_H D_2 = 2$.

Remark 3.2.2. *The author believes that Theorems 3.1.1 and 3.2.1 hold with the co-prime condition* $(a,r)=(1)$*. However, the distribution of* a/r *with* $(a,r)=(1)$ *in the fundamental region* $\mathbb{F}$ *is not uniform for some* $r\in\mathbb{Z}[\omega]\backslash\{0\}$ *and this fact makes difficulty to prove them.*

3.3 Proof of some lemmas

Before we prove Theorem 3.1.1, we first give two lemmas which will be used later.

Let δ be a positive real number. For any $a, r\in\mathbb{Z}[\omega]$ with $r\neq 0$, put

$$\mathfrak{I}_0(a,r,\delta) := \left\{z\in\mathbb{C} : \left|z-\frac{a}{r}\right|\leqslant\delta\right\}.$$

Moreover, for any $r\in\mathbb{Z}[\omega]$ with $r\neq 0$ and any closed disc $\mathfrak{I}$ in $\mathbb{C}$, we denote by $N(r,\mathfrak{I})$ (resp. $N'(r,\mathfrak{I})$) the number of $a\in\mathbb{Z}[\omega]$ satisfying $\mathfrak{I}_0\cap\mathfrak{I}\neq\phi$ (resp. $\mathfrak{I}_0(a,r,\delta)\subset\mathfrak{I}$).

Lemma 3.3.1. *Let* $\mathfrak{I}$ *be a closed disc with diameter* ζ *and* δ, η *real numbers with* $0<\delta<\zeta/4$ *and* $0<\eta<1$*. Then there exist positive constants* $c_1(d,\eta)$, $c_2(d,\eta)$ *and* $R_0(d,\eta)$*, depending only on* d *and* η*, satisfying the following: for any* $r\in\mathbb{Z}[\omega]\backslash\{0\}$ *with* $\zeta>|r|^{\eta-1}$ *and* $|r|>R_0(d,\eta)$*, we have*

$$N(r,\mathfrak{I}) \leqslant c_1(d,\eta)\zeta^2|r|^2,$$
$$N'(r,\mathfrak{I}) \geqslant c_2(d,\eta)\zeta^2|r|^2.$$

Proof. We only consider the case of $d\equiv 1 \pmod 4$. In fact, we can prove the case of $d\equiv 2,3 \pmod 4$ in the same way. Suppose $z_0\in\mathbb{C}$ is the center of $\mathfrak{I}$, i.e., $\mathfrak{I}=\{z\in\mathbb{C} : |z-z_0|\leqslant\frac{\zeta}{2}\}$. If $\mathfrak{I}_0(a,r,\delta)$ intersects $\mathfrak{I}$, then we consider the bigger disc $\mathfrak{I}'=\{z\in\mathbb{C} : |z-z_0|\leqslant\frac{\zeta}{2}+\delta\}$ and count the number of lattice points of $a\in\mathbb{Z}[\omega]$ with $a/r\in\mathfrak{I}'$ for a fixed $r\in\mathbb{Z}[\omega]\backslash\{0\}$ to estimate $N(r,\mathfrak{I})$. Let $c_1(d)=\sqrt{9-d}/2$ be the diameter and $c_2(d)=\sqrt{-d}/2$ be the area of the parallelgram $\mathbb{F}$. Then we have

$$\begin{aligned}
N(r,\mathfrak{I}) &\leqslant \frac{\pi((\frac{\zeta}{2}+\delta)+\frac{c_1(d)}{|r|})^2}{\frac{c_2(d)}{|r|^2}}\\
&\leqslant \frac{\pi}{c_2(d)}(\zeta|r|+c_1(d))^2\\
&= \frac{\pi}{c_2(d)}(\zeta^2|r|^2+2c_1(d)\zeta|r|+c_1^2(d)).
\end{aligned}$$

Since $\zeta > |r|^{\eta-1}$, $\zeta^{-1}|r|^{-1} \to 0$ as $|r|$ tends to ∞. So we see that for $|r| > R_0(d,\eta)$ with some large $R_0(d,\eta)$, there is some $c_1(d,\eta) > 0$ such that

$$N(r,\mathcal{I}) \leqslant c_1(d,\eta)\zeta^2|r|^2.$$

Similarly we count the number of lattice points in a smaller disc to estimate $N'(r,\mathcal{I})$ as follows:

$$\begin{aligned} N'(r,\mathcal{I}) &\geqslant \frac{\pi((\frac{\zeta}{2}-\delta)-\frac{c_1(d)}{|r|})^2}{\frac{c_2(d)}{|r|^2}} \\ &\geqslant \frac{\pi}{c_2(d)}(\frac{\zeta}{4}|r|-c_1(d))^2 \\ &= \frac{\pi}{c_2(d)}(\frac{\zeta^2}{16}|r|^2-\frac{c_1(d)}{2}\zeta|r|+c_1^2(d)). \end{aligned}$$

So for $|r| > R_0(d,\eta)$, there is some $c_2(d,\eta) > 0$ such that

$$N'(r,\mathcal{I}) \geqslant c_2(d,\eta)\zeta^2|r|^2.$$

$\square$

The next lemma gives the estimate for the number of two different closed discs which intersect each other described in Lemma 3.3.1.

Lemma 3.3.2. *Given a positive integer Q. For $\delta > 0$ and $a, r \in \mathbb{Z}[\omega]$ with $r \neq 0$ and $a/r \in \mathbb{F}$, put*

$$\mathcal{I}(a,r,\delta) = \left\{ z \in \mathbb{F} : \left| z - \frac{a}{r} \right| \leqslant \delta \right\}.$$

Consider

$$\mathcal{G} = \{\mathcal{I}(a,r,\delta) : (a,r) \in \Sigma, r \in \mathcal{C}\}$$

for any subset $\mathcal{C}$ of $\left\{r \in \mathcal{A} : |r|^2 \in (0,Q]\right\}$, where $\mathcal{A}$ is any infinite subset of $\mathbb{Z}[\omega]\backslash\{0\}$. Then there is some constant $k'(d) > 0$ depending on d such that

$$\left(\sum_{\substack{\mathcal{I},\mathcal{J}\in\mathcal{G} \\ \mathcal{I}\neq\mathcal{J},\mathcal{I}\cap\mathcal{J}\neq\phi}} 1 \right) \leqslant 4N_d k'(d)\delta^2 Q^2 |\mathcal{C}|^2, \tag{3.1}$$

where N_d is the number of units of $\mathbb{Q}(\sqrt{d})$.

Proof. We have

$$\begin{aligned}\left(\sum_{\substack{\mathcal{I},\mathcal{J}\in\mathcal{G}\\ \mathcal{I}\neq\mathcal{J},\mathcal{I}\cap\mathcal{J}\neq\phi}} 1\right) &\leqslant \sum_{r,s\in\mathcal{C}} \sum_{\substack{a,b\in\mathbb{Z}[\omega]\\ \frac{a}{r},\frac{b}{s}\in\mathbb{F}\\ 0<\left|\frac{a}{r}-\frac{b}{s}\right|\leqslant 2\delta}} 1 = \sum_{r,s\in\mathcal{C}} \sum_{\substack{a,b\in\mathbb{Z}[\omega]\\ \frac{a}{r},\frac{b}{s}\in\mathbb{F}\\ 0<|as-br|\leqslant\delta|rs|}} 1\\ &\leqslant \sum_{r,s\in\mathcal{C}} \sum_{\substack{a,k\in\mathbb{Z}[\omega]\\ \frac{a}{r},\frac{b}{s}\in\mathbb{F}\\ 0<|k|\leqslant 2\delta Q\\ as\equiv k \pmod{(r)}}} 1, \end{aligned} \tag{3.2}$$

for $k = as - br$. Let $U = (r, s)$ and then there are ideals R' and S' such that $(r) = UR'$ and $(s) = US'$. First, we consider the number of k with

$$U \mid (k) \quad \text{and} \quad 1 \leqslant |k|^2 \leqslant 4\delta^2 Q^2. \tag{3.3}$$

Let's denote by $T(t)$ the number of ideals whose norms are smaller than or equal to $t > 0$ and by $\mathrm{N}(\cdot)$ the norm of ideal. Put $(k) = UU'$ with an ideal U', then the number of (k) satisfying (3.3) equals to the number of U' with $\mathrm{N}(U') \leqslant 4\delta^2Q^2/\mathrm{N}(U)$, which is smaller than $T(4\delta^2Q^2/\mathrm{N}(U))$. Fix one $k \in \mathbb{Z}[\omega]$ which satisfies (3.3) and suppose that $a_0, b_0 \in \mathbb{Z}[\omega]$ and $a_1, b_1 \in \mathbb{Z}[\omega]$ are two different pairs of integers with $k = a_0 s - b_0 r = a_1 s - b_1 r$. Then we have $(a_0 - a_1)S' = (b_0 - b_1)R'$, which shows that a_0 and a_1 are in the same residue class modulo the ideal R'. Since the number of residue classes modulo the ideal R' is $\mathrm{N}(R')$ and the number of $a \in \mathbb{Z}[\omega]$ with $a/r \in \mathbb{F}$ is $|r|^2$ and these integers a are all in different residue classes modulo the ideal (r), the number of pairs of $a, b \in \mathbb{Z}[\omega]$ with $k = as - br$ is $|r|^2\mathrm{N}^{-1}(R') = \mathrm{N}(U)$ for fixed $k \in \mathbb{Z}[\omega]$. Thus we have

$$\begin{aligned}\sum_{\substack{a,k\in\mathbb{Z}[\omega]\\ \frac{a}{r},\frac{b}{s}\in\mathbb{F}\\ 0<|k|\leqslant 2\delta Q\\ as\equiv k \pmod{(r)}}} 1 &\leqslant N_d \cdot T\left(\frac{4\delta^2Q^2}{\mathrm{N}(U)}\right) \cdot \mathrm{N}(U) \leqslant 4N_d k'(d)\frac{\delta^2Q^2}{\mathrm{N}(U)} \cdot \mathrm{N}(U)\\ &= 4N_d k'(d)\delta^2 Q^2,\end{aligned}$$

with some $k'(d) > 0$. Note that N_d is always a constant. The constant $k'(d)$, depending on d, exists since the number of units in an imaginary quadratic field is finite and the sequence $\{T(n)/n\}$ converges to some constant depending on d by Theorem 1.114 in

[18]. By the above result and inequality (3.2) we have

$$\left(\sum_{\substack{\mathcal{I},\mathcal{J}\in\mathcal{G}\\ \mathcal{I}\neq\mathcal{J},\mathcal{I}\cap\mathcal{J}\neq\phi}} 1\right) \leqslant \sum_{r,s\in\mathcal{C}} 4N_d k'(d)\delta^2 Q^2 = 4N_d k'(d)\delta^2 Q^2 |\mathcal{C}|^2.$$

□

3.4 Proof of main results

Now we will give the proof of Theorem 3.1.1.

Proof of Theorem 3.1.1. First, we show (i) of Definition 2 holds for the set D. For any $\beta > 2(1+\nu)/(1+\rho)$ and any $\varepsilon > 0$, we can choose a sufficiently large $X > 0$ with

$$\frac{2}{(X)^{\frac{1+\rho}{2}}} < \varepsilon \quad \text{and} \quad \sum_{\substack{r\in\mathcal{A}\\ |r|^2>X}} \frac{2^\beta}{(|r|^2)^{(\frac{\rho\beta+\beta}{2}-1)}} < 1.$$

This is possible since $(\rho\beta+\beta)/2 - 1 > \nu$, which means

$$\sum_{r\in\mathcal{A}} \left(\frac{1}{|r|^2}\right)^{\frac{\rho\beta+\beta}{2}-1} < \infty.$$

We denote by $\{\mathcal{I}_1, \mathcal{I}_2, \cdots\}$ the collection of the discs of the form $\mathcal{I}_0(a, r, |r|^{-1-\rho})$, where $a \in \mathbb{Z}[\omega]$, $r \in \mathcal{A}$, $|r|^2 > X$, and $a/r \in \mathbb{F}$. Then the set D can be covered by the union of $\{\mathcal{I}_j\}_{j=1}^\infty$ and this satisfies condition (a) in Definition 2. Next, we have

$$\sum_{j=1}^{\infty} (\mathrm{diam}(\mathcal{I}_j))^\beta = \sum_{\substack{(a,r)\in\Sigma\\ r\in\mathcal{A}\\ |r|^2>X}} \left(\frac{2}{|r|^{1+\rho}}\right)^\beta = \sum_{\substack{r\in\mathcal{A}\\ |r|^2>X}} \frac{2^\beta}{(|r|^2)^{\frac{\rho\beta+\beta}{2}-1}} < 1,$$

which satisfies condition (b) in Definition 2. Condition (c) holds for our choice of the closed discs with $|r|^2 > X$, which satisfies

$$\mathrm{diam}(\mathcal{I}_j) = \frac{2}{|r|^{1+\rho}} < \frac{2}{X^{\frac{1+\rho}{2}}} < \varepsilon$$

for all $j \in \mathbb{N}$. Thus we see that the set D satisfies (i) of Definition 2, i.e., $\dim_H D \leqslant 2(1+\nu)/(1+\rho)$ holds.

Next, we show that the set D satisfies (ii) of Definition 2, i.e., $\dim_H D \geqslant 2(1+\nu)/(1+\rho)$. Pick some g with $0 \leqslant g \leqslant \nu$ such that

$$\sum_{r\in\mathcal{A}} (|r|^2)^{-g} = \infty.$$

Then there are infinitely many integers K satisfying

$$\sum_{\substack{r\in\mathcal{A} \\ \frac{1}{2}K \leqslant |r|^2 < K}} 1 > \frac{K^g}{\log^2 K}. \tag{3.4}$$

We show this by a contradiction. Suppose there are only finitely many rational integers of $\{K_1, K_2, ..., K_N\}$ which satisfy (3.4) with some $N \in \mathbb{N}$. Let

$$\frac{1}{2}K_0 = \max(K_1, K_2, ..., K_N).$$

Then we have

$$\sum_{\substack{r\in\mathcal{A} \\ |r|^2 < \frac{1}{2}K_0}} \left(\frac{1}{|r|^2}\right)^g < \infty.$$

For any $K \geqslant K_0$ we have

$$\sum_{\substack{r\in\mathcal{A} \\ \frac{1}{2}K \leqslant |r|^2 < K}} 1 \leqslant \frac{K^g}{\log^2 K}.$$

This shows

$$\begin{aligned}
\sum_{\substack{r\in\mathcal{A} \\ |r|^2 \geqslant \frac{1}{2}K_0}} \left(\frac{1}{|r|^2}\right)^g &= \sum_{\substack{r\in\mathcal{A} \\ \frac{1}{2}K_0 \leqslant |r|^2 < K_0}} \frac{1}{|r|^{2g}} + \sum_{\substack{r\in\mathcal{A} \\ K_0 \leqslant |r|^2 < 2K_0}} \frac{1}{|r|^{2g}} + \cdots \\
&\leqslant \left(\frac{2}{K_0}\right)^g \frac{(K_0)^g}{\log^2(K_0)} + \left(\frac{1}{K_0}\right)^g \frac{(2K_0)^g}{\log^2(2K_0)} + \cdots \\
&= 2^g \left(\frac{1}{\log^2(K_0)} + \frac{1}{\log^2(2K_0)} + \frac{1}{\log^2(2^2K_0)} + \cdots\right) \\
&= 2^g \sum_{m=0}^{\infty} \frac{1}{(k_0 + mk')^2} < \infty
\end{aligned} \tag{3.5}$$

with $k_0 = \log(K_0)$ and $k' = \log 2$. Hence we have

$$\sum_{r\in\mathcal{A}} (|r|^2)^{-g} < \infty,$$

which gives the contradiction.

Next, let $\beta < 2(1+g)/(1+\rho)$ and choose $\eta > 0$ for Lemma 3.3.1 with

$$\eta \leqslant \min\left(\frac{1}{4}(\rho - g), \frac{1}{4}\left(\frac{1+g}{1+\rho} - \frac{\beta}{2}\right)\right).$$

Choose a sequence of integers of $\{K_j\}_{j=0}^{\infty}$ satisfying the following conditions:

$$\begin{cases} \text{(i)} K_0 = 1, \\ \text{(ii)} K_1 > \max\{2R_0^2(d,\eta),\ (4N_d k'(d))^{\frac{1}{2\eta}},\ 2\cdot 4^{\frac{1}{1-\eta}},\ (\frac{8}{2^{\eta}c_2(d,\eta)})^{\frac{1}{\eta}},\ 64^{\frac{1}{1+\rho}}\}, \\ \text{(iii)} 2\log^2(2|r|^2) < |r|^{2\eta} \ \text{ for all } \ r \in \mathbb{Z}[\omega]\backslash\{0\} \ \text{ with } \ |r|^2 \geqslant K_1, \\ \text{(iv)} K_j^{1-\eta} > K_{j-1}^{1+\rho} \text{ and } K_j > 4K_{j-1} \text{ for all } j \geqslant 1, \\ \text{(v)} \displaystyle\sum_{\substack{r\in\mathcal{A} \\ \frac{1}{2}K_j \leqslant |r|^2 < K_j}} 1 > \frac{(K_j)^g}{\log^2(K_j)} \text{ and } (K_j)^g\left(1 - \frac{1}{\log^2 K_j}\right) \geqslant 2 \text{ for all } j \geqslant 1, \end{cases} \tag{3.6}$$

where $c_2(d,\eta)$ and $R_0(d,\eta)$ are from Lemma 3.3.1 and $k'(d)$ is the constant from Lemma 3.3.2. Let $D' = D \cap \mathbb{F}'$, where $\mathbb{F}'$ is a subset of $\mathbb{F}$ defined by

$$\mathbb{F}' = \left\{z \in \mathbb{C} : \left|z - \frac{1+\omega}{2}\right| \leqslant \frac{1}{4}\right\}.$$

Since $\dim_H D' \leqslant \dim_H D$, it is enough to show that $\dim_H D' \geqslant \frac{2(1+\nu)}{1+\rho}$ by checking (ii) of Definition 2. Put $\varepsilon = 2K_2^{-1/2}$ and we will show that for any sequence of closed discs of $\{\mathcal{I}_j\}_{j=1}^{\infty}$ which satisfies conditions (b) and (c) in Definition 2 does not satisfy (a), that is, if

$$\sum_{j=1}^{\infty}(\operatorname{diam}(\mathcal{I}_j))^{\beta} < 1 \tag{3.7}$$

and

$$\operatorname{diam}(\mathcal{I}_j) < \varepsilon = 2\left(\frac{1}{K_2}\right)^{\frac{1}{2}} \quad \text{for all } j \in \mathbb{N}$$

hold, then $D' \not\subset \cup_{j=1}^{\infty}\mathcal{I}_j$. We construct a collection of nested sets $\{\mathcal{J}_j\}_{j=1}^{\infty}$ with $\mathcal{J}_1 \supset \mathcal{J}_2 \supset \mathcal{J}_3 \supset \cdots$ so that $\mathcal{J} = \cap_{j=1}^{\infty}\mathcal{J}_j \subset D'$ and $\mathcal{J} \not\subset \cup_{j=1}^{\infty}\mathcal{I}_j$. Then we have $D' \not\subset \cup_{j=1}^{\infty}\mathcal{I}_j$, which completes our proof.

To do this, we define a sequence of positive real numbers $\{\varepsilon_j\}_{j=0}^{\infty}$ with $\varepsilon_j = 2(K_j)^{-\frac{1+\rho}{2}}$ for any $j \geqslant 0$. We construct the nested sets $\{\mathcal{J}_j\}_{j=1}^{\infty}$ satisfying the following four properties by induction:

(P1) $\mathcal{J}_j$ is a union of M_j disjoint closed discs with diameters $\varepsilon_j = 2(K_j)^{-\frac{1+\rho}{2}}$.

(P2) For any $\mathcal{I}_m$ with diameter between ε_j and ε_{j-1}, we have $\mathcal{I}_m \cap \mathcal{J}_j = \phi$.

(P3) For any $z \in \mathcal{J}_j$, there exist $a \in \mathbb{Z}[\omega]$ and $r \in \mathcal{A}$ with $(1/2)K_j \leqslant |r|^2 < K_j$ such

that $|z-a/r| \leqslant (K_j)^{-\frac{1+\rho}{2}}$ with $a/r \in \mathbb{F}'$;
(P4) $M_j \geqslant (K_j)^{1+g-2\eta}$.
By (P3), we have $\mathcal{J} \subset D'$. Since $\mathcal{J}_j$ is compact for all $j \in \mathbb{N}$, $\mathcal{J} = \cap_{j=1}^{\infty} \mathcal{J}_j \neq \phi$. By (P2), for any $a \in \mathcal{J}$ we have $a \notin \mathfrak{I}_j$ for all $j \in \mathbb{N}$, so $a \notin \cup_{j=1}^{\infty} \mathfrak{I}_j$ which shows $\mathcal{J} \not\subset \cup_{j=1}^{\infty} \mathfrak{I}_j$. Thus it is enough to construct $\{\mathcal{J}_j\}_{j=1}^{\infty}$ with the above four properties to show $D' \not\subset \cup_{j=1}^{\infty} \mathfrak{I}_j$.

By (3.6), we can choose a set $\mathcal{C}_1 \subset \{r \in \mathcal{A} : K_1/2 \leqslant |r|^2 < K_1\}$ such that

$$\frac{(K_1)^g}{\log^2 K_1} \leqslant |\mathcal{C}_1| \leqslant (K_1)^g, \tag{3.8}$$

where $|\mathcal{C}_1|$ denotes the cardinality of the set $\mathcal{C}_1$. Then we construct $\mathcal{J}_1$ by using the closed discs centered at $a/r \in \mathbb{F}'$ with $r \in \mathcal{C}_1$ and their radius are $\varepsilon_1/2$ which are wholly within $\mathbb{F}'$. By Lemma 3.3.1, the number of closed discs we could choose is more than $(c_2(d,\eta)/4)\sum_{r\in\mathcal{C}_1}|r|^2$. It's obvious that these closed discs all satisfy the property (P3). By the choice of ε, they also satisfy the property (P2). By Lemma 3.3.2 for $\delta = \varepsilon_1/2$, the number of pairs of discs intersecting to each other is at most $4N_d k'(d)(K_1)^{1-\rho}|\mathcal{C}_1|^2$. Remove one disc from each pairs of discs intersecting to each other and denote by M_1 the number of the left closed discs such that property (P1) holds. Now we confirm that M_1 satisfies the property (P4). Indeed we have

$$\begin{aligned}
M_1 &\geqslant \frac{c_2(d,\eta)}{4}\sum_{r\in\mathcal{C}_1}|r|^2 - 4N_d k'(d)(K_1)^{1-\rho}|\mathcal{C}_1|^2 \\
&> \frac{c_2(d,\eta)}{4}\sum_{r\in\mathcal{C}_1} 2(|r|^2)^{1-\eta}\log^2(2|r|^2) - 4N_d k'(d)(K_1)^{-2\eta}(K_1)^{1+g-2\eta} \\
&> \frac{2^\eta c_2(d,\eta)}{4}(K_1)^{1-\eta}\log^2(K_1)|\mathcal{C}_1| - (K_1)^{1+g-2\eta} \\
&> \left(\frac{2^\eta c_2(d,\eta)K_1^\eta}{4} - 1\right)(K_1)^{1+g-2\eta} \\
&> (K_1)^{1+g-2\eta}.
\end{aligned}$$

The above discussion implies that $\mathcal{J}_1$ can actually be constructed. Suppose $\mathcal{J}_j$ has already been constructed and now we will construct $\mathcal{J}_{j+1}$. Similarly to the choice of $\mathcal{C}_1$, we can find $\mathcal{C}_{j+1} \subset \{r \in \mathcal{A} : K_{j+1}/2 \leqslant |r|^2 < K_{j+1}\}$ which satisfies

$$\frac{(K_{j+1})^g}{\log^2 K_{j+1}} \leqslant |\mathcal{C}_{j+1}| \leqslant (K_{j+1})^g. \tag{3.9}$$

We only use the closed discs of $\{z \in \mathbb{C} : |z - a/r| \leqslant \varepsilon_{j+1}/2\}$ with $a/r \in \mathbb{F}'$ and $r \in \mathcal{C}_{j+1}$ which are wholly within $\mathcal{J}_j \subset \mathcal{J}_1 \subset \mathbb{F}'$ to construct $\mathcal{J}_{j+1}$ satisfying (P3). The steps of our construction of $\mathcal{J}_{j+1}$ are as follows:

(step 1) Choose all the closed discs $\{z \in \mathbb{F}' : |z - a/r| \leqslant \varepsilon_{j+1}/2\}$ which are wholly within $\mathcal{J}_j$.
(step 2) Remove the closed discs which intersect to each other such that all the left closed discs are all disjoint.
(step 3) Remove all the closed discs which intersect some closed discs in $\{\mathcal{I}_j\}_{j=1}^{\infty}$ whose diameter is between ε_{j+1} and ε_j.
(step 4) Confirm the number of closed discs, that is, whether $M_{j+1} \geqslant (K_{j+1})^{1+g-2\eta}$ or not.
(step 5) If (step 4) satisfies property (P4), then define $\mathcal{J}_{j+1}$ as the union of the left closed discs.

Let $\zeta = \varepsilon_j$ and $\delta = \varepsilon_{j+1}/2 = (K_{j+1})^{-\frac{1+\rho}{2}}$. By our choice of $\{K_j\}$ in (3.6) we have $\delta < (4K_j)^{-\frac{1+\rho}{2}} < (1/4)\varepsilon_j = (1/4)\zeta$. From our choice of K_j in (3.6) with $(K_{j+1})^{1-\eta} > (K_j)^{1+\rho}$, the number of closed discs which are wholly within $\mathcal{J}_j$ is more than

$$c_2(d,\eta)M_j\varepsilon_j^2 \sum_{r\in\mathcal{C}_{j+1}} |r|^2 \tag{3.10}$$

by using Lemma 3.3.1. By Lemma 3.3.2 for $\delta = \varepsilon_{j+1}/2$, we have that the number of pairs of closed discs which intersect to each other is less than

$$4N_d k'(d)\left(\frac{\varepsilon_{j+1}}{2}\right)^2 (K_{j+1})^2|\mathcal{C}_{j+1}|^2 = 4N_d k'(d)(K_{j+1})^{1-\rho}|\mathcal{C}_{j+1}|^2. \tag{3.11}$$

Define

$$\mathcal{F}_j = \{\mathcal{I} \in \{\mathcal{I}_j\}_{j=1}^{\infty} : \varepsilon_{j+1} \leqslant \operatorname{diam}(\mathcal{I}) < \varepsilon_j\},$$

and put

$$\mathcal{F}_j^{(1)} = \{\mathcal{I} \in \mathcal{F}_j : 2\left(\frac{1}{K_{j+1}}\right)^{\frac{1-\eta}{2}} \leqslant \operatorname{diam}(\mathcal{I}) < \varepsilon_j\},$$

$$\mathcal{F}_j^{(2)} = \{\mathcal{I} \in \mathcal{F}_j : \varepsilon_{j+1} \leqslant \operatorname{diam}(\mathcal{I}) < 2\left(\frac{1}{K_{j+1}}\right)^{\frac{1-\eta}{2}}\}.$$

By Lemma 3.3.1, we see that the number of closed discs in $\mathcal{J}_{j+1}$ which intersect some closed discs in $\mathcal{F}_j$ is less than

$$\sum_{\mathcal{I}\in\mathcal{F}_j^{(1)}} \sum_{r\in\mathcal{C}_{j+1}} c_1(d,\eta)(\operatorname{diam}(\mathcal{I}))^2|r|^2 + \sum_{\mathcal{I}\in\mathcal{F}_j^{(2)}} \sum_{r\in\mathcal{C}_{j+1}} \left(\frac{5}{2}(\operatorname{diam}(\mathcal{I}) + \varepsilon_{j+1})|r|\right)^2 \tag{3.12}$$

for some $c_1(d,\eta) > 0$. From (3.7) we see

$$\begin{aligned}\sum_{\mathcal{J}\in\mathcal{F}_j^{(1)}} c_1(d,\eta)(\operatorname{diam}(\mathcal{J}))^2 \sum_{r\in\mathcal{C}_{j+1}} |r|^2 &= \sum_{\mathcal{J}\in\mathcal{F}_j^{(1)}} c_1(d,\eta)(\operatorname{diam}(\mathcal{J}))^{2-\beta}(\operatorname{diam}(\mathcal{J}))^{\beta} \sum_{r\in\mathcal{C}_{j+1}} |r|^2 \\ &\leqslant c_1(d,\eta)(\varepsilon_j)^{2-\beta}\left(\sum_{\mathcal{J}\in\mathcal{F}_j^{(1)}} (\operatorname{diam}(\mathcal{J}))^{\beta}\right)\left(\sum_{r\in\mathcal{C}_{j+1}} |r|^2\right) \\ &< c_1(d,\eta)(\varepsilon_j)^{2-\beta}\left(\sum_{r\in\mathcal{C}_{j+1}} |r|^2\right).\end{aligned}$$

Since

$$(\varepsilon_j)^{2-\beta} = 2^{2-\beta}\left(\frac{1}{K_j}\right)^{\frac{(1+\rho)}{2}(2-\beta)} < 4\left(\frac{1}{K_j}\right)^{\frac{(1+\rho)}{2}(2-\beta)-4\eta\rho} \leqslant 4\left(\frac{1}{K_j}\right)^{\rho-g+4\eta},$$

we have

$$\sum_{\mathcal{J}\in\mathcal{F}_j^{(1)}} c_1(d,\eta)(\operatorname{diam}(\mathcal{J}))^2 \sum_{r\in\mathcal{C}_{j+1}} |r|^2 < 4c_1(d,\eta)(K_j)^{g-\rho-4\eta}\left(\sum_{r\in\mathcal{C}_{j+1}} |r|^2\right). \tag{3.13}$$

The estimate of the second sum in (3.12) is

$$\begin{aligned}\sum_{\mathcal{J}\in\mathcal{F}_j^{(2)}} \sum_{r\in\mathcal{C}_{j+1}} \left(\frac{5}{2}(\operatorname{diam}(\mathcal{J})+\varepsilon_{j+1})|r|\right)^2 &< 100\left(\frac{1}{K_{j+1}}\right)^{\frac{(1-\eta)}{2}(2-\beta)} \sum_{r\in\mathcal{C}_{j+1}} |r|^2 \\ &< 100\left(\frac{1}{K_{j+1}}\right)^{3\eta}\left(\sum_{r\in\mathcal{C}_{j+1}} |r|^2\right),\end{aligned} \tag{3.14}$$

since

$$\frac{(1-\eta)}{2}(2-\beta) - 3\eta = 1 - 4\eta - \frac{\beta}{2} + \frac{\beta\eta}{2} \geqslant \frac{\rho-g}{1+\rho} + \frac{\beta\eta}{2} > 0.$$

Finally, we estimate M_{j+1} in (step 4). From (3.10), (3.11), (3.13), and (3.14) we have

$$\begin{aligned}M_{j+1} &\geqslant c_2(d,\eta)M_j\varepsilon_j^2 \sum_{r\in\mathcal{C}_{j+1}} |r|^2 - 4N_d k'(d)(K_{j+1})^{1-\rho}|\mathcal{C}_{j+1}|^2 \\ &- 4c_1(d,\eta)(K_j)^{g-\rho-4\eta}\left(\sum_{r\in\mathcal{C}_{j+1}} |r|^2\right) - 100(K_{j+1})^{-3\eta}\left(\sum_{r\in\mathcal{C}_{j+1}} |r|^2\right) \qquad (3.15)\end{aligned}$$

$$\tag{3.16}$$

By (3.6) and $4\eta \leqslant \rho - g$, we have

$$\begin{aligned}
4N_d k'(d)(K_{j+1})^{1-\rho}|\mathcal{C}_{j+1}|^2 &\leqslant 4N_d k'(d)(K_{j+1})^{1-\rho}(K_{j+1})^{2g} \\
&\leqslant 4N_d k'(d)(K_{j+1})^{1+g-4\eta} \\
&= 4N_d k'(d)(K_{j+1})^{g}(K_{j+1})^{1-\eta}(K_{j+1})^{-3\eta} \\
&= 4N_d k'(d)\frac{1}{2}\cdot 2^{1-\eta}\cdot (K_{j+1})^{-3\eta}\frac{(K_{j+1})^g}{\log^2(K_{j+1})} \\
&\cdot\ 2\left(\frac{1}{2}K_{j+1}\right)^{1-\eta}\log^2(K_{j+1}) \\
&< 4N_d k'(d)(K_{j+1})^{-3\eta}\left(\sum_{r\in\mathcal{C}_{j+1}}|r|^2\right). \qquad (3.17)
\end{aligned}$$

From (3.15) and (3.17), we get

$$\begin{aligned}
M_{j+1} &> \left(\sum_{r\in\mathcal{C}_{j+1}}|r|^2\right) \\
&\cdot \left(c_2(d,\eta)M_j\varepsilon_j^2 - 4c_1(d,\eta)(K_j)^{g-\rho-4\eta} - (4N_d k'(d)+100)(K_{j+1})^{-3\eta}\right) \\
&\geqslant \left(\sum_{r\in\mathcal{C}_{j+1}}|r|^2\right) \\
&\cdot \left(4c_2(d,\eta)(K_j)^{g-\rho-2\eta} - 4c_1(d,\eta)(K_j)^{g-\rho-4\eta} - (4N_d k'(d)+100)(K_{j+1})^{-3\eta}\right).
\end{aligned}$$

Here, we can add some more conditions to our choice of $\{K_j\}$ for all $j \geqslant 1$:

$$K_1 > \left(\frac{2c_1(d,\eta)}{c_2(d,\eta)} + (2N_d k'(d)+50)\right)^{\frac{1}{\eta}}, \qquad (3.18)$$

$$K_j > \left(\frac{(K_{j-1})^{\rho+2\eta-g}}{2c_2(d,\eta)}\right)^{\frac{1}{\eta}}. \qquad (3.19)$$

By (3.19), we have

$$(K_{j+1})^{-3\eta} < (K_j)^{-2\eta}\cdot(K_{j+1})^{-\eta} < (K_j)^{-2\eta}\cdot 2c_2(d,\eta)(K_j)^{g-\rho-2\eta}$$

and then we see

$$\begin{aligned}
M_{j+1} &> \left(\sum_{r\in\mathcal{C}_{j+1}}|r|^2\right) 2c_2(d,\eta)(K_j)^{g-\rho-2\eta} \\
&\cdot \left(2 - \left(\frac{2c_1(d,\eta)}{c_2(d,\eta)} + (2N_d k'(d)+50)\right)(K_j)^{-2\eta}\right) \\
&> \left(\sum_{r\in\mathcal{C}_{j+1}}|r|^2\right) 2c_2(d,\eta)(K_j)^{g-\rho-2\eta}.
\end{aligned}$$

Since

$$\begin{aligned}\left(\sum_{r\in\mathcal{C}_{j+1}}|r|^2\right) &> \sum_{r\in\mathcal{C}_{j+1}} 2\cdot(|r|^2)^{1-\eta}\log^2(2|r|^2)\\ &\geqslant \frac{(K_{j+1})^g}{\log^2(K_{j+1})}\cdot 2\left(\frac{1}{2}K_{j+1}\right)^{1-\eta}\log^2(K_{j+1})\\ &= 2^\eta(K_{j+1})^{1-\eta+g}\end{aligned}$$

and by (3.19), we have

$$(K_j)^{g-\rho-2\eta} > \frac{(K_{j+1})^{-\eta}}{2c_2(d,\eta)}.$$

This gives

$$\begin{aligned}M_{j+1} &> 2^\eta(K_{j+1})^{1-\eta+g}\cdot 2c_2(d,\eta)\cdot\frac{(K_{j+1})^{-\eta}}{2c_2(d,\eta)}\\ &= 2^\eta(K_{j+1})^{1+g-2\eta} > (K_{j+1})^{1+g-2\eta},\end{aligned}$$

which satisfies the property (P4). So we can actually construct $\mathcal{J}_{j+1}$ from $\mathcal{J}_j$. By this construction, we have $D' \not\subset \cup_{j=1}^{\infty}\mathcal{I}_j$. Thus we see that $\dim_H D' \geqslant \frac{2(1+\nu)}{1+\rho}$, which completes the proof of Theorem 3.1.1. □

Next, we give the proof of Theorem 3.2.1 by using Theorem 3.1.1.

Proof of Theorem 3.2.1. From Theorem 1.1 in [6] we see if $\Psi((r)) = \mathcal{O}(|r|^{-1})$ then D_2 has the full Lebesgue measure which also means $\dim_H D_2 = 2$. Thus it is enough to consider only the case where $\Psi((r)) = \mathcal{O}(|r|^{-1})$ doesn't hold, i.e., there are infinitely many $r \in \mathbb{Z}[\omega]\backslash\{0\}$ such that $\Psi((r)) > |r|^{-1}$. Let's define

$$\hat{\Psi}((r)) = \begin{cases}\Psi((r)), & \text{if } \Psi((r)) > |r|^{-1},\\ 0, & \text{otherwise,}\end{cases}$$

and put $\mathcal{A}' = \{r \in \mathbb{Z}[\omega]\backslash\{0\} : \hat{\Psi}((r)) \neq 0\}$. If $\sum_{r\in\mathcal{A}'}\Phi((r))\hat{\Psi}^2((r))|r|^{-2}$ converges, then $\sum_{r\notin\mathcal{A}'}\Phi((r))\Psi^2((r))|r|^{-2}$ diverges. By Theorem 1.1 in [6] again, the Hausdorff dimension of the set D_2 is 2 for the sequence $\{\Psi((r))\}$. Now let's consider the case of $\sum_{r\in\mathcal{A}'}\Phi((r))\hat{\Psi}^2((r))|r|^{-2}$ diverges. In this case, it is enough to prove it with $\{\hat{\Psi}((r))\}$ instead of $\{\Psi((r))\}$.

We restrict $\hat{\Psi}((r)) \leqslant 1$ for all $r \in \mathbb{Z}[\omega]\backslash\{0\}$ without loss of generality. For any given $\varepsilon > 0$, let

$$\mathcal{A}(m) = \{r \in \mathcal{A}' : |r|^{-(m+1)\varepsilon} < \hat{\Psi}((r)) \leqslant |r|^{-m\varepsilon}\}$$

for $0 \leqslant m < [\varepsilon^{-1}]$ and put

$$\mathcal{A}([\varepsilon^{-1}]) = \{r \in \mathcal{A}' : |r|^{-1} < \hat{\Psi}((r)) \leqslant |r|^{-[\varepsilon^{-1}]\varepsilon}\}.$$

Since

$$\sum_{r \in \mathcal{A}'} \hat{\Psi}^2((r)) = \sum_{m=0}^{[\frac{1}{\varepsilon}]} \sum_{r \in \mathcal{A}(m)} \hat{\Psi}^2((r)) = \infty,$$

there is at least one m with $0 \leqslant m \leqslant [\varepsilon^{-1}]$ such that

$$\sum_{r \in \mathcal{A}(m)} \hat{\Psi}^2((r)) = \infty \tag{3.20}$$

with $|\mathcal{A}(m)| = \infty$. By (3.20) and $(|r|^2)^{-m\varepsilon} \leqslant 1$, there exists a sequence of $\{\mathcal{B}_n\}$ of pairwise disjoint nonempty subsets of $\mathcal{A}(m)$ satisfying the following conditions:
(1) $\mathcal{A} = \cup_{n=1}^{\infty} \mathcal{B}_n$.
(2) Let $n \leqslant n'$ be any positive integers. Then, for any $r \in \mathcal{B}_n$ and $r' \in \mathcal{B}_{n'}$, we have $|r| \leqslant |r'|$.
(3) For any positive integer n, we have

$$1 \leqslant \sum_{r \in \mathcal{B}_n} \left(\frac{1}{|r|^2}\right)^{m\varepsilon} \leqslant 2.$$

For any $n \in \mathbb{N}$, put $\eta_n = 2^{-n}$. Then there exists $k_n \in \mathbb{N}$ such that

$$\sum_{r \in \mathcal{B}_k} \left(\frac{1}{|r|^2}\right)^{m\varepsilon + \eta_n} < \frac{1}{2^{n-1}} \tag{3.21}$$

holds for any $k \geqslant k_n$. So we have a sequence $\{k_n\}$ with $k_1 < k_2 < k_3 < \cdots$ which satisfies (3.21). Put $\mathcal{B} = \cup_{j=1}^{\infty} \mathcal{B}_{k_j}$, then $\mathcal{B}$ is an infinite subset of $\mathcal{A}(m)$ and obviously satisfies

$$\sum_{r \in \mathcal{B}} \left(\frac{1}{|r|^2}\right)^{m\varepsilon} = \infty.$$

For any $h > m\varepsilon$, there exists some $n_0 \in \mathbb{N}$ with $h > m\varepsilon + \eta_n$ for all $n \geqslant n_0$, which shows

$$\begin{aligned}
\sum_{r \in \mathcal{B}} \left(\frac{1}{|r|^2}\right)^{h} &< \sum_{r \in \cup_{j=1}^{n_0-1} B_{k_j}} \left(\frac{1}{|r|^2}\right)^{h} + \sum_{j=n_0}^{\infty} \sum_{r \in \mathcal{B}_{k_j}} \left(\frac{1}{|r|^2}\right)^{m\varepsilon + \eta_j} \\
&< \sum_{r \in \cup_{j=1}^{n_0-1} B_{k_j}} \left(\frac{1}{|r|^2}\right)^{h} + \sum_{j=n_0}^{\infty} \frac{1}{2^{j-1}} < \infty.
\end{aligned}$$

Thus $\mathcal{B}$ is an infinite subset of $\mathcal{A}(m)$ satisfying

$$\begin{cases} \sum_{r\in\mathcal{B}} \left(\frac{1}{|r|^2}\right)^h = \infty, & \text{if } h \leqslant m\varepsilon, \\ \sum_{r\in\mathcal{B}} \left(\frac{1}{|r|^2}\right)^h < \infty, & \text{if } h > m\varepsilon. \end{cases}$$

Let

$$D_2' = \left\{ z \in \mathbb{F} : \left| z - \frac{a}{r} \right| < \frac{1}{|r|^{1+(m+1)\varepsilon}} \ \textit{has infinitely many } (a,r) \in \Sigma \ \textit{with } r \in \mathcal{B} \right\}.$$

Then we have $\dim_H D_2 \geqslant \dim_H D_2'$ since $D_2' \subset D_2$. Let $\nu = m\varepsilon$ and $\rho = (m+1)\varepsilon$. By Theorem 3.1.1 we see

$$2 \geqslant \dim_H D_2 \geqslant \dim_H D_2' = \frac{2(1+\nu)}{1+\rho} = \frac{2(1+m\varepsilon)}{1+m\varepsilon+\varepsilon} > 2 - 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $\dim_H D_2 = 2$. □

Acknowledgements

I would like to acknowledge my supervisor Prof. Hitoshi Nakada for his very important suggestions and advices. I also would like to thank Prof. Valérie Berthé, Prof. Taka-aki Tanaka and Prof. Takeshi Katsura for their useful advice.

Bibliography

[1] V. Beresnevich, G. Harman, A. Haynes and S. Velani, *The Duffin-Schaeffer conjecture with extra divergence II*, Math. Z. 275 (2013), No.1-2 127–133.

[2] F. Bernstein, *Über eine Anwendung der Mengenlehre auf ein aus der Theorie der säkularen Störungen herrührendes Problem*, Math. Ann. 71 (1912), 417–39.

[3] A. S. Besicovitch, *Sets of fractional dimension IV: On rational approximation to real numbers*, J. London Math. Soc. 9 (1934), 126–131.

[4] E. Borel, *Les probabilités denombrables et leurs applications arithmétiques*, Rend. Circ. Math. Palermo 27 (1909), 247–71.

[5] J. W. S. Cassels, *Some metrical theorems in Diophantine approximation I*, Proc. Cambridge Phil. Soc. 46 (1950), 209–18.

[6] Z. Chen, *On metric theory of Diophantine approximations for complex numbers*, Acta Arithmetica, 170 (2015), No.1, 27–46.

[7] R. J. Duffin and A. C. Schaeffer, *Khintchine's problem in metric Diophantine approximation*, Duke Math. J. 8 (1941), 243–255.

[8] P. Erdös, *On the distribution of convergents of almost all real numbers*, J. Number Theory 2 (1970), 425–441.

[9] L. R. Ford, *Rational approximations to irrational complex numbers*, Trans. Amer. Math. Soc. 19 (1918), 1–42.

[10] P. X. Gallagher, *Approximation by reduced fractions*, J. Math. Soc. Japan 13 (1961), 342–5.

[11] G. Harman, *Metric Number Theory*, London Mathematical Society Monographs New Series, Oxford Science Publications, New York, No.18, 1998, 262 pp.

[12] H. Hasse, *Vorlesungen über Zahlentheorie*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Zweite Auflage, Berlin, Band 59, 1964, 389 pp.

[13] A. Haynes, A. Pollington and S. Velani, *The Duffin-Schaeffer conjecture with extra divergence*, Math. Ann. 353 (2012), No.2 259–273.

[14] A. Hurwitz, *Über die Entwicklung komplexer Grössen in Kettenbrüche*, Acta Math. 11 (1887-88), 187–200.

[15] A. Hurwitz, *Über die angenäherte Darstellung der Irrationalzahlen durch rationale Brüche*, Math. Ann. 46 (1891), 279–84.

[16] V. Jarnik, *Diophantische Approximationen und hausdorffisches*, Mass. Mat. Sb. 36 (1929), 371–382.

[17] A. Khintchine, *Einige Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der Diophantischen Approximationen*, Math. Ann. 92 (1924), 115–125.

[18] H. Koch, *Algebraic Number Theory*, Encyclopaedia of Mathematical Sciences, Springer-Verlag, Berlin, Vol.62, 1997, 84 pp.

[19] E. Landau, *Neuer Beweis des Primzahlsatzes und Beweis des Primidealsatzes*, Math. Ann. 56 (4) (1903), 645–670.

[20] H. Nakada, *The metrical theory of complex continued fractions*, Acta Arith. 56 (1990), No.4 279–289.

[21] H. Nakada and G. Wagner, *Duffin-Schaeffer Theorem of Diophantine Approximation for Complex Numbers*, Asterisque 198-199-200 (1991) 259–263.

[22] O. Perron, *Über die Approximation einer komplexen Zahl durch Zahlen des Körpers K(i)*, Math. Ann. 103 (1930), 533–544; II, Math. Ann. 105 (1931), 160–164.

[23] A.D. Pollington and R.C. Vaughan, *The k-dimensional Duffin and Schaeffer conjecture*, Journal de Théorie des Nombres de Nordeaux, 1 (1989), 81–87.

[24] M. Rosen, *A Generalization of Mertens' Theorem*, J. Ramanujan Math. Soc. 14 (1999), No.1 1–19.

[25] W. M. Schmidt, *Diophantine Approximation*, Lecture notes in Mathematics, Springer-Verlag, Berlin, Vol.785, 1996

[26] D. Sullivan, *Disjoint spheres, approximation by imaginary quadratic numbers and the logarithm law for geodesics*, Acta Arith. 149 (1983), 215–239.

[27] T. Takagi, *Daisûteki Seisûron. Gaisetsu oyobi Ruitairon. (Japanese) [Algebraic number theory. Generalities and class field theory.]*, Iwanami Shoten, Tokyo, 1971, 28 pp.

[28] J. D. Vaaler, *On the metric theory of diophantine approximation*, Pacific journal of mathematics 76 (1978), No.2 527–539.