A Thesis for the Degree of Ph.D. in Science

Explicit Representations of Locally Risk-minimizing Hedging Strategy for Lévy Markets by Malliavin Calculus

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Preface

Locally risk-minimizing (LRM, for short) is a well-known hedging method for contingent claims in a quadratic way for incomplete financial markets. Theoretical aspects of LRM have been developed to a high degree (see e.g., Schweizer [45] and [46]). LRM has an intimate relationship with Föllmer-Schweizer decomposition (FS decomposition, for short), which is a kind of orthogonal decomposition of a random variable into a stochastic integration and an orthogonal martingale. The necessity of researches on its explicit representations has been increasing. However, it is generally very difficult to derive an explicit expression for the locally risk-minimizing hedge. In this thesis, we obtain explicit representations of LRM for incomplete market models whose asset price process is described by a solution to a stochastic differential equation (SDE, for short) driven by a Lévy process, as a typical framework of incomplete market models. In particular, we use Malliavin calculus for canonical Lévy processes to achieve our purpose. Especially, we adopt a Clark-Ocone type formula under change of measure (COCM) for canonical Lévy processes. The Clark-Ocone (CO) formula is an explicit martingale representation of functionals of Brownian motions (Lévy processes) in terms of Malliavin derivatives. Girsanov transformations versions of this theorem are Clark-Ocone type formulas under change of measure. Since many applications in mathematical finance require representations of random variables with respect to risk neutral martingale measure, the theorem was studied by many people (see introduction of Chapter 3).

For our purpose, we develop and review Malliavin calculus for canonical Lévy processes. We review related topics of Malliavin calculus for canonical Lévy processes and we show some formulas to show the COCM for canonical Lévy processes, such as closability of Malliavin derivatives, chain rules for Malliavin derivative and commutation formulas for integrals and the Malliavin derivative. By using these results, we derive a COCM for canonical Lévy processes.

We next derive an LRM for Lévy markets by using these results. We first focus on deriving a representation of FS decomposition under some mild conditions by using the martingale representation theorem. In order to compute its explicit expressions, we use Malliavin calculus. Especially, we will formulate representations of LRM including Malliavin derivatives of the claim to hedge. We also derive formulas on representations of LRM for three typical options such as call options, Asian options and lookback options.

In summary, main contribution of this thesis is sixfold as follows:

1. deriving some calculation tools such as commutation formula for the Lebesgue integral and the Malliavin derivative and chain rules for Malliavin derivative.
2. formulating a Clark-Ocone type formula under change of measure for canonical Lévy processes.

3. deriving versions of the Poincaré inequality for Lévy functionals (with respect to $\mathbb{P}^*$) and the logarithmic Sobolev inequality (with respect to both $\mathbb{P}^*$ and $\mathbb{P}$).
4. formulating representations of LRM with Malliavin derivatives for Lévy markets.
5. illustrating how to calculate Malliavin derivatives for non-smooth functions of a random variable, and the running maximum of processes by using approximation methods.
6. introducing concrete representations of LRM of call options, Asian options and lookback options for Lévy markets.

This thesis is organized as follows. Chapter 2 deals with a short review of Classical Malliavin calculus. In Chapter 3, basic notions and some preliminaries of mathematical finance and (L)RM are given. Chapter 4 deals with a Malliavin calculus for Lévy processes and a Clark-Ocone type formula under change of measure for canonical Lévy processes. In Chapter 5, we obtain explicit representations of LRM for Lévy markets.

Contents

Chapter 1

Introduction

In this thesis, we consider the local risk minimization problem which is a very well-known problem in mathematical finance. Especially, we obtain explicit representations of LRM for Lévy markets by using Malliavin calculus for Lévy processes.

The Malliavin calculus (stochastic calculus of variations) is an infinite-dimensional differential calculus on the Wiener space, which was first introduced by Paul Malliavin in the 70's (see Malliavin [31]). The purpose of this calculus was to prove the results on existence and smoothness of densities of solutions to stochastic differential equations driven by a Brownian motion. This theory was developed by Bismut, Kusuoka, Shigekawa, Stroock, Watanabe and others (see, e.g., Shigekawa [47] and references therein). At the beginning, Malliavin calculus was not very popular due to its technical difficulties. However, in modern times, it is one of the most famous theories in probability. There are many applications of Malliavin calculus in many fields (see e.g., Nualart [33] and Di Nunno [20]). In Chapter 2, we give a short review of classical Malliavin calculus.

The representations of functionals of Brownian motions (or Lévy processes) by stochastic integrals are important results in Probability theory. They has been widely studied (see, e.g., survey paper by Davis [16]). In particular, the Clark-Ocone (CO) formula is an explicit martingale representation of functionals of Brownian motions in terms of Malliavin derivatives. If an L^2-random variable F has certain regularity in the Malliavin sense, we have

$$F = \mathbb{E}[F] + \int_0^T \mathbb{E}[D_t F|\mathcal{F}_t] dW_t,$$

where W is a Brownian motion and $D_t F$ is the classical Malliavin derivative. This formula was shown by Clark, Ocone and Haussmann [13, 14, 23, 36]. A white noise version of the CO formula was proved by Aase et al. [1]. This formula has various applications. For example, the log-Sobolev and Poincare inequalities are obtained in Capitaine et al. [11]. In the application to mathematical finance, its representation of an optimal portfolio is given by this formula (see e.g., Ocone and Karatzas [35]).

Malliavin calculus for Lévy processes has been also widely studied (see, e.g., Di Nunno [20], Delong [17], Ishikawa [25] and their references). This theory was at first motivated by study about existence and smoothness of densities of solutions to stochastic differential equations driven by Lévy processes as classical Malliavin calculus. Later, Malliavin calculus for Lévy processes has been also applied to mathematical finance theory in incomplete markets. In incomplete markets, the CO formula for Lévy processes is one of the most useful formula to get representation of an optimal portfolio just as the cases complete markets. The CO formula for Lévy processes has been also studied. Løkka [30] got a CO formula for functionals of pure jump Lévy processes. A white noise version of the CO formula for functionals of pure jump Lévy was proved by Di Nunno et al. [19]. We know that one for general L^2-Lévy functionals also holds (see Benth et al. [9], Delong[17] and Chapter 4 of this thesis).

Because many applications in mathematical finance require representation formula with respect to risk neutral martingale measure, CO formulas under Girsanov transformations were studied by many people. First, Ocone and Karatzas [35] showed a Clark-Ocone type formula under change of measure (COCM) for Brownian motions:

$$F = \mathbb{E}_{\mathbb{P}^*}[F] + \int_0^T \mathbb{E}_{\mathbb{P}^*}\left[D_t F - F\int_0^T D_t u_s dW_s^{\mathbb{P}^*}\middle|\mathcal{F}_t\right] dW_t^{\mathbb{P}^*}.$$

They also applied it to get an optimal portfolio of Brownian market. A white noise version of it was proved by Okur [37] and she also derived an explicit representation of hedging strategy of digital option for Brownian market. Huehne [24] got a COCM for pure jump Lévy processes and derived an optimal portfolio. Later, Di Nunno et al. [20] and Okur [38] also introduced a white noise version of COCM for Lévy processes by using white noise theory. In this thesis, we also derive a COCM for Lévy processes:

$$\begin{aligned} F = {}& \mathbb{E}_{\mathbb{P}^*}[F] + \sigma\int_0^T \mathbb{E}_{\mathbb{P}^*}\left[D_{t,0}F - FK_t\middle|\mathcal{F}_{t-}\right] dW_t^{\mathbb{P}^*} \\ & + \int_0^T\int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[F(H_{t,z}^* - 1) + zH_{t,z}^* D_{t,z}F|\mathcal{F}_{t-}]\tilde{N}^{\mathbb{P}^*}(dt,dz). \end{aligned}$$

We precisely define K_t and $H_{t,z}^*$, and give sufficient conditions for this formula in section 4.5. However, note that their results are different from our results. In our results, we use different settings and derive different representation. By using this result, we obtain log-Sobolev and Poincare type inequalities for Lévy functionals. For that purpose, we adapted Malliavin calculus for Lévy processes based on Geiss and Laukkarinen [22] and Solé et al. [49]. Moreover, we show some formulas to show the main theorem, such as chain rule for Malliavin derivative and commutation formulas for integrals and the Malliavin derivative. By using σ-finiteness of Lévy measure (see e.g., Applebaum [3]), we prove it. Moreover, we applied it to LRM in Chapter 5.

The quadratic criterion of local risk-minimization is one of the most famous concepts of hedging in incomplete markets. At the beginning, Föllmer and Sondermann [21] introduced the risk-minimizing (RM, for short) hedging strategies for contingent claims, written on a one-dimensional, square-integrable discounted risky asset S which is a martingale under the original probability measure $\mathbb{P}$. Later, Schweizer [43] showed that RM dose not always exist in the semi-martingale case. Therefore, Schweizer [44] introduced the concept of locally risk-minimizing hedging strategies to hedge claims for the case that the discounted risky asset is a semi-martingale. See survey papers Pham, Schweizer and, Vandaele and Vanmaele [39, 45, 53]. In Chapter 3, we review a basic notions and some preliminaries of mathematical finance and (L)RM.

However, the theory does not give a method of obtaining a concrete representation. Hence, the necessity of researches on its explicit representations has been increasing.

From this insight, we obtain explicit representations of LRM for incomplete market models whose asset price process is described by a solution to a stochastic differential equation driven by a Lévy process. To achieve our purpose, we use Malliavin calculus for Lévy processes. In Chapter 5, we deal with explicit representations of LRM by using Malliavin calculus for Lévy processes.

Chapter 2

A short review of classical Malliavin calculus

2.1 Classical Malliavin derivative

In this chapter, we review classical Malliavin calculus, based on Di Nunno et al. [20]. Let $T > 0$ be a finite time horizon, $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{t\in[0,T]})$ a one-dimensional Wiener space on $[0, T]$; and W its coordinate mapping process, that is, a one-dimensional standard Brownian motion with $W_0 = 0$. Let $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ be the canonical filtration completed for $\mathbb{P}$. Let $L^2_{T,\lambda,n}$ denote the set of product measurable, deterministic functions $h : ([0, T])^n \to \mathbb{R}$ satisfying

$$\|h\|^2_{L^2_{T,\lambda,n}} := \int_{([0,T])^n} |h(t_1, \cdots, t_n)|^2 dt_1 \cdots dt_n < \infty,$$

where λ is Lebesgue measure on $[0, T]$. For $n \in \mathbb{N}$ and $h_n \in L^2_{T,\lambda,n}$, we denote

$$I_n(h_n) := \int_{([0,T])^n} h(t_1, \cdots, t_n) dW_{t_1} \cdots dW_{t_n}.$$

It is easy to see that $\mathbb{E}[I_0(h_0)] = h_0$ and $\mathbb{E}[I_n(h_n)] = 0$, for $n \geq 1$. Moreover, this integral has the usual properties (see Section 1.1 of Di Nunno et al. [20]):

Proposition 2.1.1 *1. For $n \geq 1, f \in L^2_{T,\lambda,n}$, we obtain, $I_n(f) = I_n(\tilde{f})$, where $\tilde{f}$ is the symmetrization of f :*

$$\tilde{f}(t_1, \cdots, t_n) = \frac{1}{n!} \sum_{\pi\in\mathcal{D}_n} f(t_{\pi(1)}, \cdots, t_{\pi(n)}),$$

where, $\mathcal{D}_n$ is the set of permutations of $\{1, 2, \cdots, n\}$.

2. *For $n \geq 1, a, b \in \mathbb{R}, f, g \in L^2_{T,\lambda,n}$, we get: $I_n(af + bg) = aI_n(f) + bI_n(g)$.*
3. *For $m, n \geq 1, f \in L^2_{T,\lambda,n}, g \in L^2_{T,\lambda,m}$, are symmetric in the n pairs $t_i, 1 \leq i \leq n$, that is $f = \tilde{f}$ and $g = \tilde{g}$, then, we have*

$$\mathbb{E}[I_n(f)I_m(g)] = n!\mathbf{1}_{(n=m)}\langle f, g\rangle_{L^2_{T,\lambda,n}}.$$

In this setting, we introduce the following chaos expansion (see Theorem 1.10 in Di Nunno et al. [20]).

Theorem 2.1.2 *Any $\mathcal{F}$-measurable square integrable random variable F on the canonical space has a unique representation*

$$F = \sum_{n=0}^{\infty} I_n(h_n), \mathbb{P}-\text{a.s.}$$

with functions $h_n \in L^2_{T,\lambda,n}$ that are symmetric in the n pairs $t_i, 1 \leq i \leq n$ and we have the isometry

$$\mathbb{E}[F^2] = \sum_{n=0}^{\infty} n! \|h_n\|^2_{L^2_{T,\lambda,n}}.$$

By using the chaos expansion, we can define the following:

Definition 2.1.3 *(1) Let $\mathbb{D}^{1,2}_W$ denote the set of $\mathcal{F}$ -measurable random variables $F \in L^2(\mathbb{P})$ with the representation $F = \sum_{n=0}^{\infty} I_n(h_n)$ satisfying*

$$\sum_{n=1}^{\infty} n n! \|h_n\|^2_{L^2_{T,\lambda,n}} < \infty.$$

(2) Let $F \in \mathbb{D}^{1,2}_W$. Then the Malliavin derivative $DF : \Omega \times [0,T] \to \mathbb{R}$ of a random variable $F \in \mathbb{D}^{1,2}_W$ is a stochastic process defined by

$$D_t F := \sum_{n=1}^{\infty} n I_{n-1}(h_n(t,\cdot)), \quad \text{valid for } \lambda-\text{a.e. } t \in [0,T], \mathbb{P} - a.s.$$

We next establish the following fundamental result (see, Theorem 3.3 in Di Nunno et al. [20]).

Proposition 2.1.4 (The closability of operator D) *Let $F \in L^2(\mathbb{P})$ and $F_k \in \mathbb{D}^{1,2}_W, k \in \mathbb{N}$ such that*

1. *$\lim_{k\to\infty} F_k = F$ in $L^2(\mathbb{P})$,*
2. *$\{D_t F_k\}_{k=1}^{\infty}$ converges in $L^2(\lambda \times \mathbb{P})$.*

Then, $F \in \mathbb{D}^{1,2}_W$ and $\lim_{k\to\infty} D_t F_k = D_t F$ in $L^2(\lambda \times \mathbb{P})$.

We next introduce chain rules for the Malliavin derivative (see Theorem 3.5 in Di Nunno et al. [20], Proposition 1.2.4 in Nualart [33] and Lemma A.1 in Ocone and Karatzas [35] respectively).

Proposition 2.1.5 *1. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a C^1-function with bounded derivative. If $F \in \mathbb{D}^{1,2}_W$, then, $\varphi(F) \in \mathbb{D}^{1,2}_W$ and*

$$D_t \varphi(F) = \varphi'(F) D_t F \text{ for } \lambda-\text{a.e. } t \in [0,T], \ \mathbb{P}-\text{a.s.}$$

holds.

2. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function with Lipschitz constant K and $F \in \mathbb{D}^{1,2}_W$. Then, $\varphi(F) \in \mathbb{D}^{1,2}_W$. Moreover, there exists a random variable G bounded by K such that*

$$D_t\varphi(F) = GD_tF \text{ for } \lambda\text{−a.e. } t \in [0,T],\ \mathbb{P}\text{−a.s.}$$

3. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a C^1-function and assume that $\varphi(F) \in L^2(\mathbb{P})$, $F \in \mathbb{D}^{1,2}_W$ and $\varphi'(F)D_tF \in L^2(\lambda \times \mathbb{P})$. Then, $\varphi(F) \in \mathbb{D}^{1,2}_W$ and*

$$D_t\varphi(F) = \varphi'(F)D_tF \text{ for } \lambda\text{−a.e. } t \in [0,T],\ \mathbb{P}\text{−a.s.}$$

holds.

Next proposition shows that the derivative operator D_t has the local property on the space $\mathbb{D}^{1,2}_W$ (see e.g., Proposition 1.3.16 in Nualart [33]).

Proposition 2.1.6 *For any $F \in \mathbb{D}^{1,2}_W$, we have $\mathbf{1}_{\{F=0\}}D_tF = 0$, (t,ω)-a.e.*

By using Theorems 2.1.5, 2.1.4 and Proposition 2.1.6, we can derive the following:

Theorem 2.1.7 *For any $F \in \mathbb{D}^{1,2}_W$, $K \in \mathbb{R}$ and λ-a.e. $t \in [0,T]$, we have $(F-K)^+ \in \mathbb{D}^{1,2}_W$ and*

$$D_t(F-K)^+ = \mathbf{1}_{\{F>K\}}D_tF$$

where $x^+ = \max(x,0)$.

Proof. We take a mollifier function φ which is a C^∞-function from $\mathbb{R}$ to $[0,\infty)$ with $\operatorname{supp}(\varphi) \subset [-1,1]$ and $\int_{-\infty}^{\infty}\varphi(x)dx = 1$. We denote $\varphi_n(x) := n\varphi(nx)$ and $f_n(x) := \int_{-\infty}^{\infty}(y-K)^+\varphi_n(x-y)dy$ for any $n \geq 1$. Noting that

$$f_n(x) = \int_{-\infty}^{\infty}\left(x - \frac{y}{n} - K\right)^+\varphi(y)dy = \int_{-\infty}^{n(x-K)}\left(x - \frac{y}{n} - K\right)\varphi(y)dy,$$

we have $f_n'(x) = \int_{-\infty}^{n(x-K)}\varphi(y)dy$, so that $f_n \in C^1$ and $|f_n'| \leq 1$, that is, f_n is Lipschitz continuous with constant 1. Thus, Proposition 2.1.5 implies that, for any $n \geq 1$, $f_n(F) \in \mathbb{D}^{1,2}_W$ and

$$D_tf_n(F) = f_n'(F)D_tF \tag{2.1.1}$$

In addition, noting that

$$\begin{aligned}|f_n(x) - (x-K)^+| &= \left|\int_{-1}^{1}\left\{\left(x - \frac{y}{n} - K\right)^+ - (x-K)^+\right\}\varphi(y)dy\right| \\ &\leq \frac{1}{n}\int_{-1}^{1}|y|\varphi(y)dy \leq \frac{1}{n}\end{aligned} \tag{2.1.2}$$

for any $x \in \mathbb{R}$, we have $\lim_{n\to\infty}\mathbb{E}[|f_n(F) - (F-K)^+|^2] = 0$. Thus, from the view of Proposition 2.1.4, all we have to do is to make sure that $D_tf_n(F)$ converges to

$$\mathbf{1}_{\{F>K\}}D_{t,0}F =: I_\infty$$

in $L^2(\lambda \times \mathbb{P})$ as n tends to ∞.

First of all, we have

$$\lim_{n\to\infty} f_n'(x) = \begin{cases} \int_{-\infty}^{0} \varphi(y)dy & \text{if } x = K, \\ 1 & \text{if } x > K, \\ 0 & \text{if } x < K, \end{cases}$$

from which we obtain $\lim_{n\to\infty} f_n'(F) = \mathbf{1}_{\{F>K\}} + \mathbf{1}_{\{F=K\}} \int_{-\infty}^{0} \varphi(y)dy$. By (2.1.1), (2.1.2) and Proposition 2.1.6, we have $\lim_{n\to\infty} D_t f_n(F) = I_\infty$ in $\lambda \times \mathbb{P}$-a.e., and

$$\begin{aligned} &|D_t f_n(F) - I_\infty| \\ &\le |f_n'(F)D_tF - \mathbf{1}_{\{F>K\}}D_tF| \\ &\le 2|D_tF| \in L^2(\lambda \times \mathbb{P}). \end{aligned}$$

Thus, the dominated convergence theorem provides that $D_t f_n(F) \to I_\infty$ in $L^2(\lambda \times \mathbb{P})$. $\square$

2.2 The Skorohod integral and the Malliavin derivative

In this section, we consider the Skorohod integral and commutation of integration and the Malliavin differentiability. First we introduce the following classes.

Definition 2.2.1 *(1) $\mathbb{L}_W^{1,2}$ denotes the space of $G : [0,T] \times \Omega \to \mathbb{R}$ satisfying*

1. $G_s \in \mathbb{D}_W^{1,2}$ *for a.e.* $s \in [0,T]$,
2. $E\left[\int_{[0,T]} |G_s|^2 ds\right] < \infty$,
3. $E\left[\int_{[0,T]\times\mathbb{R}} \int_0^T |D_tG_s|^2 dsdt\right] < \infty$.

(2) Recall that any function $u \in L^2(\lambda \times \mathbb{P})$ has a chaotic representation

$$u_t = \sum_{n=0}^{\infty} I_n(h_n(\cdot, t)),$$

where $h_n \in L^2_{T,\lambda,n+1}$ is symmetric in the first n pairs of variables. Denoting by $\hat{h}_n$ the symmetrization of h_n with respect to all $n+1$ pairs of variables, we define

$$\mathrm{Dom}_\delta^W := \left\{ u \in L^2(\lambda \times \mathbb{P}) \middle| \sum_{n=0}^{\infty} (n+1)! \|\hat{h}_n\|^2_{L^2_{T,\lambda,n+1}} < \infty \right\}.$$

(3) Let $u \in \mathrm{Dom}_\delta^W$. Then the Skorohod integral δ^W with respect to the W of a process u : $\Omega \times [0,T] \to \mathbb{R}$ is defined as

$$\delta^W(u) = \sum_{n=0} I_{n+1}(\hat{h}_n),\ \mathbb{P}\text{−a.s.}$$

The Skorohod integral δ^W has the following properties:

Proposition 2.2.2 (1) Duality formula (Theorem 3.14 in Di Nunno et al. [20])
A process $u \in L^2(\lambda \times \mathbb{P})$ belongs to Dom_δ^W if and only if there exists a constant C such that for all $F \in \mathbb{D}_W^{1,2}$,

$$\left| \mathbb{E}\left[\int_{[0,T]} u_s D_s F ds \right] \right| \leq C(\mathbb{E}[F^2])^{1/2}.$$

If $u \in \mathrm{Dom}_\delta^W$, then $\delta^W(u)$ is the element of $L^2(\mathbb{P})$ characterized by

$$\mathbb{E}[\delta(u)F] = \mathbb{E}\left[\int_{[0,T]} u_s D_s F ds \right]$$

for any $F \in \mathbb{D}_W^{1,2}$.
(2) Differentiability of δ^W (Theorem 3.18 in Di Nunno et al. [20])
Let $u \in \mathbb{L}_W^{1,2}$ such that $D_t u \in \mathrm{Dom}_\delta^W$ for all $t \in [0,T]$, λ-a.e. and assume that $\delta^W(D_t u) \in L^2(\lambda \times \mathbb{P})$. Then $\delta^W(u) \in \mathbb{D}_W^{1,2}$ and

$$D_t \delta^W(u) = u_t + \delta(D_t u)$$

for all $t \in [0,T]$, λ-a.e.
(3) (Theorem 2.9 in Di Nunno et al. [20])
Let $u \in L^2(\lambda \times \mathbb{P})$ be predictable. Then, $u \in \mathrm{Dom}_\delta^W$ and

$$\delta^W(u) = \int_{[0,T]} u_s dW_s.$$

Hence, we can see that the Skorohod integral is an extension of the Itô integral.

We next discuss the commutation relation of the stochastic integral with the Malliavin derivative.

Proposition 2.2.3 (Corollary 3.19 of Di Nunno et al. [20]) *Let $G : \Omega \times [0,T]$ be a predictable process with*

$$\mathbb{E}\left[\int_{[0,T]} |G_s|^2 ds \right] < \infty.$$

Then

$$G \in \mathbb{L}_W^{1,2} \text{ if and only if } \int_{[0,T]} G_s dW_s \in \mathbb{D}_W^{1,2}.$$

Furthermore, if $\int_{[0,T]} G_s dW_s \in \mathbb{D}_W^{1,2}$, then, for λ -a.e. $t \in [0,T]$, we have

$$D_t \int_{[0,T]} G_s dW_s = G_t + \int_{[0,T]} D_t G_s dW_s, \quad \mathbb{P}\text{−a.s.},$$

and $\int_{[0,T]} D_t G_s dW_s$ is a stochastic integral in Itô sense.

By using the Malliavin derivative and the Skorohod integral, we can derive the following (see e.g., Proposition 2.2 in Nualart [34]):

Proposition 2.2.4 (Existence of density) *Let F be a random variable such that $F \in \mathbb{D}^{1,2}_W$. Assume that $\frac{D_t F}{\|D_\cdot F\|^2_{L^2(\lambda)}} \in \mathrm{Dom}^W_\delta$. Then the law of F has a continuous and bounded density function given by*

$$f(x) = \mathbb{E}\left[\mathbf{1}_{\{F>x\}}\delta^W\left(\frac{D_t F}{\|D_\cdot F\|^2_{L^2(\lambda)}}\right)\right], x \in \mathbb{R}.$$

2.3 The Clark-Ocone formula and the Girsanov theorem

2.3.1 The Clark-Ocone type formula

We next present an explicit form of the martingale representation formula by using Malliavin calculus (see e.g., Theorem 4.1 in Di Nunno et al. [20]).

Proposition 2.3.1 (The Clark-Ocone type formula) *Let $F \in \mathbb{D}^{1,2}_W$. Then, we have*

$$F = \mathbb{E}[F] + \int_0^T \mathbb{E}[D_t F|\mathcal{F}_t]dW_t.$$

By using the Clark-Ocone formula, we can derive the following (see Capitaine et al. [11]):

Proposition 2.3.2 1. Poincare's inequality
Let $F \in \mathbb{D}^{1,2}_W$. Then, we have

$$\mathbb{E}[(F - \mathbb{E}[F])^2] \leq \int_0^T \mathbb{E}[|D_t F|^2]dt.$$

2. Logarithmic Sobolev inequality
Let $F \in \mathbb{D}^{1,2}_W$ and $F \geq \varepsilon$ for some $\varepsilon > 0$. Then, we obtain

$$\mathbb{E}[F^2 \log F^2] - \mathbb{E}[F^2]\log \mathbb{E}[F^2] \leq 2\int_0^T \mathbb{E}[|D_t F|^2]dt.$$

2.3.2 Girsanov theorem

We recall the Girsanov theorem for Brownian motions (see, e.g., Section 4.1 of Di Nunno et al. [20]).

Theorem 2.3.3 *Let $u_s, s \in [0, T]$, be predictable processes such that $\int_0^T u_s^2 ds < \infty$,* a.s. *Moreover we denote*

$$Z_t := \exp\left(-\int_0^t u_s dW_s - \frac{1}{2}\int_0^t u_s^2 ds\right), t \in [0, T].$$

Define a measure $\mathbb{P}^*$ on $\mathcal{F}_T$ by

$$d\mathbb{P}^*(\omega) = Z_T(\omega)d\mathbb{P}(\omega),$$

and we assume that $Z(T)$ satisfies the Novikov condition, that is,

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T u_s^2 ds\right)\right] < \infty.$$

Then $\mathbb{E}[Z_T] = 1$ and hence $\mathbb{P}^*$ is a probability measure on $\mathcal{F}_T$. Furthermore if we denote

$$dW_t^{\mathbb{P}^*} := u_t dt + dW_t,$$

then $W^{\mathbb{P}^*}(\cdot)$ is a standard Brownian motion under $\mathbb{P}^*$.

2.4 Clark-Ocone formula under change of measure

In this section, we introduce a Clark-Ocone formula under change of measure. Throughout this section, under the same setting as Theorem 2.3.3, we assume the following.

Assumption 2.4.1 *1. $u, u^2 \in \mathbb{L}_W^{1,2}$; and $2u_s D_t u_s \in L^2(\lambda \times \mathbb{P})$ for a.e. $s \in [0, T]$.*
2. $Z_T \in L^2(\mathbb{P})$; and $Z_T D_t \log Z_T \in L^2(\lambda \times \mathbb{P})$.
3. $F \in \mathbb{D}_W^{1,2}$ with $FZ_T \in L^2(\mathbb{P})$; and $Z_T D_t F + F D_t Z_T \in L^2(\lambda \times \mathbb{P})$.

We next introduce a Clark-Ocone type formula under change of measure (see e.g., Theorem 4.5 in Di Nunno et al. [20]).

Theorem 2.4.2

$$F = \mathbb{E}_{\mathbb{P}^*}[F] + \int_0^T \mathbb{E}_{\mathbb{P}^*}\left[D_t F - F\int_0^T D_t u_s dW_s^{\mathbb{P}^*} \middle| \mathcal{F}_t\right] dW_t^{\mathbb{P}^*}, \text{ a.s.}$$

holds.

Chapter 3

Basic concepts of mathematical finance and LRM

3.1 Basic notions of mathematical finance

3.1.1 Basic notions of mathematical finance

In this section, we give an overview of basic concepts in mathematical finance theory (see also e.g., Klebaner [27], Lamberton and Lapeyre [29] and Miyahara [32]). In mathematical finance theory, pricing and hedging of a contingent claim is central problem, where a contingent claim on an asset is a contract that allows purchase or sale of this asset in the future on terms that are specified in the contract. We consider a financial market being composed of one risk-free asset (e.g. money market, cash or bond) and one risky asset (e.g. stock) with finite time horizon T. We now introduce a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\in[0,T]})$, where the filtration is supposed to be right-continuous, complete and $\mathcal{F}_0$ is trivial. The fluctuation of the risky asset is assumed to be given by a semi-martingale $S = (S)_{t\in[0,T]}$. This process is adapted and has cádlág paths. The risk-less asset price process is given by $B = (B_t)_{t\in[0,T]}, B_0 = 1$. We assume that B_t is continuous and of finite variation. Let ξ_t and η_t denote the amount of units of the risky asset and the risk-free asset an investor holds at time t. The market value of the portfolio at time t is given by $V_t = \xi_t S_t + \eta_t B_t$.

Definition 3.1.1 *A portfolio (ξ_t, η_t) is called self-financing if*

$$dV_t = \xi_t dS_t + \eta_t dB_t,$$

i.e.

$$V_t = V_0 + \int_0^t \xi_u dS_u + \int_0^t \eta_u dB_u.$$

We can see the following:

Theorem 3.1.2 (Theorem 11.11 in Klebaner [27]) *A portfolio (ξ_t, η_t) is self-financing if and only if, the discounted value process $\frac{V_t}{B_t}$ is a stochastic integral with respect to the discounted price* *process*

$$\frac{V_t}{B_t} = V_0 + \int_0^t \xi_u d\tilde{S}_u,$$

where $\tilde{S} = \frac{S_t}{B_t}$.

We next define arbitrage opportunity.

Definition 3.1.3 *A self-financing portfolio* (ξ_t, η_t) *is called an arbitrage opportunity if* V_t *satisfies the following conditions:* $V_0 = 0$, $\mathbb{P}(V_T \geq 0) = 1$ *and* $\mathbb{P}(V_T > 0) > 0$.

If there exists an equivalent martingale measure, i.e. a probability measure $\mathbb{P}^*$ equivalent to the original probability measure $\mathbb{P}$ such that the discounted price process $\tilde{S}$ is a (local) martingale under $\mathbb{P}^*$, then the market model contains no arbitrage opportunities. Absence of arbitrage is basis for mathematical finance theory. We next consider pricing of claims.

Definition 3.1.4

1. *A predictable and self-financing strategy* (ξ_t, η_t) *is called admissible if* $\sqrt{\int_0^t \xi_u^2 d[\tilde{S}, \tilde{S}]_u}$ *is finite and locally integrable for* $t \in [0, T]$. *Moreover,* V_t/B_t *is non-negative* $\mathbb{P}^*$*-martingale.*
2. *Let* $F \geq 0$ *be a contingent claim. It is attainable (or redundant) if it is integrable and there exists an admissible trading strategy such that* $V_T = F$.

We can derive the following (see e.g., Theorem 11.13 in Klebaner [27]):

Theorem 3.1.5 *The price* P_t *at time* t *of an attainable claim* F *is given by the value of an admissible replicating portfolio* V_t, *and*

$$P_t = \mathbb{E}_{\mathbb{P}^*}\left[\frac{B_t}{B_T} F | \mathcal{F}_t\right].$$

Theorem 3.1.6 *Let* F *be a integrable contingent claim and let* $N_t = \mathbb{E}_{\mathbb{P}^*}\left[\frac{F}{B_T} | \mathcal{F}_t\right]$ *for* $t \in [0, T]$. *Then* F *is attainable if and only if* N_t *can be represented in the form*

$$N_t = N_0 + \int_0^t \tilde{\xi}_u d\tilde{S}_u$$

for some predictable process $\tilde{\xi}$. *Moreover,* $V_t/B_t = N_t$ *is the same for any admissible portfolio that replicates* F.

We next consider completeness of a market model.

Definition 3.1.7 *A market model is complete if any integrable claim is attainable, in other words, can be replicated by a self-financing portfolio.*

Next theorem is called second fundamental theorem of mathematical finance (see e.g., Theorem 11.15 in Klebaner [27]).

Theorem 3.1.8 *The following are equivalent:*

1. *The market model is complete*
2. *The equivalent martingale measure $\mathbb{P}^*$ that makes $\tilde{S}_t = \frac{S_t}{B_t}$ into a martingale is unique.*

If our market is complete, then, we can get price of claim uniquely. Moreover, Theorem 3.1.6 implies that

$$V_t/B_t = V_0 + \int_0^t \tilde{\xi}_u d\tilde{S}_u$$

and

$$F = V_0 + \int_0^T \tilde{\xi}_t d\tilde{S}_t.$$

Therefore, we can see that the claim can be replicated at time T with initial investment V_0 and the following strategy at time t :

$$(\tilde{\xi}_t, V_0 + \int_0^t \tilde{\xi}_u d\tilde{S}_u - \tilde{\xi}\tilde{S}_u).$$

We next deal the Black-Scholes-Merton model as typical model of complete market.

3.1.2 Black-Scholes-Merton model

The Black-Scholes-Merton model (BSM model, in short) is the most popular and fundamental model in mathematical finance. Let $T > 0$ be a finite time horizon, $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{t\in[0,T]})$ a one-dimensional Wiener space on $[0, T]$; and W its coordinate mapping process, that is, a one-dimensional standard Brownian motion with $W_0 = 0$. Let $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ be the canonical filtration completed for $\mathbb{P}$. In the Black-Scholes-Merton model, we assume that the market consists of one risky asset and one risk-less asset. The fluctuation of the risky asset is assumed to be given by the following stochastic differential equation (SDE):

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad S_0 > 0,$$

where μ is a real number (called mean rate of return), σ is a positive real number (called volatility). The solution of the SDE is given by

$$S_t = S_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right].$$

The risk-less asset price process $(B_t)_{t\in[0,T]}$ is given by

$$B_t = e^{rt}, r \geq 0,$$

where r is risk-less interest rate. The discounted stock process is given by

$$\tilde{S}_t = \frac{S_t}{B_t} = S_0 \exp\left[\left(\mu - r - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right]$$

or

$$
\begin{aligned}
d\tilde{S}_t &= \tilde{S}_t[(\mu - r)dt + \sigma dW_t] \\
&= \sigma \tilde{S}_t d\left[W_t + \frac{\mu - r}{\sigma} t\right], \quad S_0 > 0.
\end{aligned}
$$

We now let $u := \frac{\mu - r}{\sigma}$ and

$$
Z_t = \exp\left(-uW_t - \frac{1}{2}u^2 t\right), t \in [0, T].
$$

Then, Theorem 2.3.3 implies that $\mathbb{E}[Z_T] = 1$ and $W_t^{\mathbb{P}^*} = W_t + ut$ is a Brownian motion under $\mathbb{P}^*$ with $d\mathbb{P}^* = Z_T d\mathbb{P}$. Moreover, Theorem 11.16 in Klebaner [27] shows that $\mathbb{P}^*$ is unique equivalent martingale measure that makes $\tilde{S}_t = \frac{S_t}{B_t}$ into a martingale. Hence, BSM model is complete market with non-arbitrage by Theorem 3.1.8. Theorem 3.1.5 implies that the price of a claim F is given by

$$
P_t = e^{-r(T-t)} \mathbb{E}_{\mathbb{P}^*}\left[F | \mathcal{F}_t\right].
$$

Let ξ_t and η_t denote the amount of units of the risky asset and the risk-free asset an investor holds at time t and assume that ξ_t and η_t are adapted processes satisfying $\int_0^T \xi_t^2 dt, \int_0^T |\eta_t| dt < \infty$ a.s. The market value of the discounted self-financing replication portfolio at time t is given by

$$
\begin{aligned}
\tilde{V}_t &= \xi_t \tilde{S}_t + \eta_t \\
&= \xi_0 S_0 + \eta_0 + \int_0^t \xi_u d\tilde{S}_u \\
&= V_0 + \int_0^t \xi_u \tilde{S}_u \sigma dW_u^{\mathbb{P}^*}
\end{aligned}
$$

and

$$
e^{-rT} F = e^{-rT} V_T = V_0 + \int_0^T \xi_t \tilde{S}_t \sigma dW_t^{\mathbb{P}^*}.
$$

We next derive the Black-Scholes-Merton formula, that is, theoretical price of the European call option $(S_T - K)^+$, where $K > 0$ is a strike price at T. European call option is a contract that gives its holder the right (but not the obligation) to buy the risky asset with value S_T at the maturity time T at a fixed price K. By Theorem 3.1.5, we can get the initial price of the European call option

$$
P_0 = e^{-rT} \mathbb{E}_{\mathbb{P}^*}\left[(S_T - K)^+\right] = S_0 N(d_1) - e^{-rT} K N(d_2)
$$

where $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}y^2} dy$,

$$
d_1 = \frac{\log(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},
$$

and

$$d_2 = \frac{\log(S_0/K) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}.$$

We also get the price P_t at time t of the European call option

$$P_t = e^{-r(T-t)}\mathbb{E}_{\mathbb{P}^*}\left[(S_T - K)^+ | \mathcal{F}_t\right] = S_t N(d_1(t)) - e^{-r(T-t)} K N(d_2(t))$$

where

$$d_1(t) = \frac{\log(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2(t) = \frac{\log(S_t/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}.$$

We next derive hedging strategy of the European call option by using classical Malliavin calculus. We first check conditions of Assumption 2.4.1.

1. Since u is constant, hence $D_t u = 0$. Therefore, we can see that $u, u^2 \in \mathbb{L}_W^{1,2}$; and $2uD_t u \in L^2(\lambda \times \mathbb{P})$ hold.
2. It is easy to see that $Z_T \in L^2(\mathbb{P})$. Moreover, we have $D_t \log Z_T = -u$. Therefore $Z_T D_t \log Z_T \in L^2(\lambda \times \mathbb{P})$ holds. Moreover, Proposition 2.1.5 implies that $D_t Z_T = -u Z_T$.
3. Since $S_T \in L^2(\mathbb{P})$ and $D_t \log S_T = \sigma$, we can see that $S_T D_t \log S_T \in L^2(\lambda \times \mathbb{P})$ holds. Hence, Proposition 2.1.5 implies that $S_T \in \mathbb{D}_W^{1,2}$ and $D_t S_T = \sigma S_T$. Moreover, Theorem 2.1.7 shows that $(S_T - K)^+ \in \mathbb{D}_W^{1,2}$ and

   $$D_t(S_T - K)^+ = \mathbf{1}_{\{S_T > K\}} D_t S_T = \mathbf{1}_{\{S_T > K\}} \sigma S_T.$$

   Since $|D_t(S_T - K)^+| \leq \sigma S_T$ and $|(S_T - K)^+| \leq S_T + K$, we can see that $Z_T D_t(S_T - K)^+ + (S_T - K)^+ D_t Z_T \in L^2(\lambda \times \mathbb{P})$.

Hence, we can apply Theorem 2.4.2 to $e^{-rT}(S_T - K)^+$. Theorem 2.4.2 implies that

$$\begin{aligned}
&e^{-rT} V_T \\
&= V_0 + \int_0^T \xi_t \tilde{S}_t \sigma dW_t^{\mathbb{P}^*} \\
&= e^{-rT}(S_T - K)^+ \\
&= e^{-rT}\mathbb{E}_{\mathbb{P}^*}[(S_T - K)^+] + e^{-rT}\int_0^T \mathbb{E}_{\mathbb{P}^*}\left[D_t(S_T - K)^+ - (S_T - K)^+ \int_0^T D_t u dW_s^{\mathbb{P}^*} \middle| \mathcal{F}_t\right] dW_t^{\mathbb{P}^*} \\
&= e^{-rT}\mathbb{E}_{\mathbb{P}^*}[(S_T - K)^+] + e^{-rT}\int_0^T \mathbb{E}_{\mathbb{P}^*}\left[\mathbf{1}_{\{S_T > K\}} \sigma S_T \middle| \mathcal{F}_t\right] dW_t^{\mathbb{P}^*}.
\end{aligned}$$

Hence, we obtain

$$\xi_t \tilde{S}_t \sigma = e^{-rT}\mathbb{E}_{\mathbb{P}^*}\left[\mathbf{1}_{\{S_T > K\}} \sigma S_T \middle| \mathcal{F}_t\right].$$

Therefore the portfolio is given by

$$\xi_t = \frac{e^{-r(T-t)}}{S_t}\mathbb{E}_{\mathbb{P}^*}\left[\mathbf{1}_{\{S_T>K\}}S_T\middle|\mathcal{F}_t\right] = N(d_1(t)).$$

In this subsection, we saw that the BSM model is a complete market model. However, it is said that the real market is incomplete in general. In the incomplete case, there are many equivalent martingale measure and there exists some claims that is impossible to replicate. Therefore, we can not determine price and hedging strategy of claim uniquely. Hence, we have to choose a suitable hedging method for incomplete market model. We present in this thesis (locally) risk-minimizing that is a very well-known hedging method for contingent claims in a quadratic way for incomplete financial markets.

3.2 Risk minimization

In this section, we review basic notions of risk minimization. Föllmer and Sondermann [21] introduced the risk-minimizing (RM, for short) hedging strategies for non-redundant contingent claims, written on a one-dimensional, square-integrable discounted risky asset S which is a martingale under the original measure $\mathbb{P}$. We now introduce a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\in[0,T]})$, where the filtration is supposed to be right-continuous, complete and $\mathcal{F}_0$ is trivial. The goal of RM is to minimize the variance of future costs: $R_t = \mathbb{E}[(C_T - C_t)^2|\mathcal{F}_t]$, where C_t means cost process which will defined later.

Definition 3.2.1 *1. Θ_S denotes the space of all $\mathbb{R}$-valued predictable processes ξ satisfying*

$$\mathbb{E}[\int_0^T \xi_t^2 d\langle S\rangle_t] < \infty$$

.

2. *An L^2-strategy is given by a pair $\varphi = (\xi, \eta)$, where $\xi \in \Theta_S$ and η is an adapted process such that $V(\varphi) := \xi S + \eta$ is a right continuous process with $\mathbb{E}[V_t^2(\varphi)] < \infty$ for every $t \in [0, T]$. Note that ξ_t (resp. η_t) represents the amount of units of the risky asset (resp. the risk-free asset) an investor holds at time t.*
3. *For $F \in L^2(\mathbb{P})$, the process $C^F(\varphi)$ defined by $C_t^F(\varphi) := F1_{\{t=T\}} + V_t(\varphi) - \int_0^t \xi_s dS_s$ is called the cost process of $\varphi = (\xi, \eta)$ for F.*
4. *For contingent claim $F \in L^2(\mathbb{P}; \mathcal{F}_T)$, we call F-admissible if $V_T = 0$.*

We know that the following: if S is a martingale, the claim $F \in L^2(\mathbb{P})$ has the following decomposition:

$$F = \mathbb{E}[F] + \int_0^T \xi_s^* dS_s + L_T^F,$$

where $\xi^* \in \Theta_S$ and L^F is a square-integrable martingale orthogonal to S with $L_0^F = 0$. We call this decomposition the Galtchouk-Kunita-Watanabe decomposition (see Kunita and Watanabe [28]). Moreover, the unique F-admissible risk-minimizing strategy φ^* is given by

$$\varphi^* = (\xi^*, \mathbb{E}[F|\mathcal{F}_t] - \xi_t^* S_t)$$

for all $t \in [0, T]$ (see e.g., section 2 of Vandaele and Vanmaele [52]).

In the case S is a semi-martingale under $\mathbb{P}$, we could still look for risk-minimizing strategies φ with $V_T(\varphi) = 0$. Unfortunately, there is bad news (see Proposition 3.1 of Schweizer [45]):

Proposition 3.2.2 *If S is not a (local) $\mathbb{P}$-martingale, a contingent claim F admits in general no risk-minimizing strategy φ with $V_T(\varphi) = 0$. $\mathbb{P}$-a.s.*

Hence, we consider the concept of locally risk-minimizing hedging strategies to hedge claims in next section.

3.3 Local risk minimization

In this section, we review basic notions of local risk minimization. Schweizer [43] proved that RM dose not always exist in the semi-martingale case. Therefore, Schweizer [44] introduced the concept of locally risk-minimizing hedging strategies to hedge claims for the case that the discounted risky asset is a semi-martingale. We can see streams of research of the LRM by survey papers (see, e.g., Pham, Schweizer and Vandaele and Vanmaele [39, 45, 53]) and we can also see that theoretical aspects of LRM has been developed to a high degree.

We now consider a incomplete financial market being composed of one risk-free asset and one risky asset with finite time horizon T. For simplicity, we assume that the interest rate of the market is given by 0, that is, the price of the risk-free asset is 1 at all times. The fluctuation of the risky asset is assumed to be given by a semi-martingale S on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\in[0,T]})$, where the filtration is supposed to be right-continuous, complete and $\mathcal{F}_0$ is trivial. The semi-martingale S has the following decomposition

$$S = S_0 + M + A,$$

where M a square-integrable martingale for which $M_0 = 0$, and with A a predictable process of finite variation $|A|$. We also assume the following assumption.

Assumption 3.3.1 *S satisfying the so-called structure condition (SC, for short). That is S satisfies*

$$\left\| [M]_T^{1/2} + \int_0^T |dA_s| \right\|_{L^2(\mathbb{P})} < \infty, \tag{3.3.1}$$

A is absolutely continuous with respect to $\langle M \rangle$ with a density λ satisfies $\mathbb{E}[\langle \int \lambda dM \rangle] < \infty$, we can rewrite the canonical decomposition as $S = S_0 + M + \int \lambda d\langle M \rangle$. Thirdly, the mean-variance trade-off process $K_t := \int_0^t \lambda_s^2 d\langle M \rangle_s$ is finite, that is, K_T is finite $\mathbb{P}$-a.s.

We define locally risk-minimizing (LRM, for short) for a contingent claim $F \in L^2(\mathbb{P})$. We first define L^2-strategy and cost process.

Definition 3.3.2 1. *Θ_S denotes the space of all $\mathbb{R}$-valued predictable processes ξ satisfying*

$$\mathbb{E}\left[\int_0^T \xi_t^2 d\langle M \rangle_t + \left(\int_0^T |\xi_t dA_t|\right)^2\right] < \infty$$

.

2. *An L^2-strategy is given by a pair $\varphi = (\xi, \eta)$, where $\xi \in \Theta_S$ and η is an adapted process such that $V(\varphi) := \xi S + \eta$ is a right continuous process with $\mathbb{E}[V_t^2(\varphi)] < \infty$ for every $t \in [0, T]$. Note that ξ_t (resp. η_t) represents the amount of units of the risky asset (resp. the risk-free asset) an investor holds at time t.*
3. *For $F \in L^2(\mathbb{P})$, the process $C^F(\varphi)$ defined by $C_t^F(\varphi) := F1_{\{t=T\}} + V_t(\varphi) - \int_0^t \xi_s dS_s$ is called the cost process of $\varphi = (\xi, \eta)$ for F.*

We next introduce the definition of a small perturbation.

Definition 3.3.3 (Small Perturbation) *A trading strategy $\Delta = (\delta, \varepsilon)$ is called a small perturbation if it satisfies the following:*

1. *δ is bounded,*
2. *$\int_0^T |\delta_t dA_t|$ is bounded,*
3. *$\delta_T = \varepsilon_T = 0$.*

For any subinterval $(s, t]$ of $[0, T]$, we define the small perturbation

$$\Delta|_{(s,t]} := (\delta \mathbf{1}_{(s,t]}, \varepsilon \mathbf{1}_{[s,t)}).$$

We also define partitions $\tau = (t_i)_{0 \leq i \leq N}$ of the interval $[0, T]$. A partition of $[0, T]$ is a finite set $\tau = \{t_0, t_1, \cdots, t_k\}$ of times with $0 = t_0 < t_1 < \cdots < t_k = T$ and the mesh size of τ is $|\tau| := \max_{t_i, t_{i+1} \in \tau}(t_{i+1} - t_i)$. A sequence $(\tau_n)_{n \in \mathbb{N}}$ is called increasing if $\tau_n \subseteq \tau_{n+1}$ for all n and it tends to the identity if $\lim_{n \to \infty} |\tau_n| = 0$. We next define the locally risk-minimizing.

Definition 3.3.4 (Locally Risk-minimizing) *For a trading strategy φ, a small perturbation Δ and a partition τ of $[0, T]$ the risk quotient $r^\tau[\varphi, \Delta]$ is defined as follows:*

$$r^\tau(\varphi, \Delta) := \sum_{t_i, t_{i+1} \in \tau} \frac{R_{t_i}(\varphi + \Delta|_{(t_i, t_{i+1}]}) - R_{t_i}(\varphi)}{\mathbb{E}[\langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i} | \mathcal{F}_{t_i}]} \mathbf{1}_{(t_i, t_{i+1}]},$$

where $R_{t_i} = \mathbb{E}[(C_T - C_{t_i})^2 | \mathcal{F}_{t_i}]$. A trading strategy φ is called locally risk-minimizing if

$$\liminf_{n \to \infty} r^{\tau_n}(\varphi, \Delta) \geq 0$$

$\mathbb{P} \otimes \langle M \rangle$-a.e. on $\Omega \times [0, T]$ for every small perturbation Δ and every increasing sequence $(\tau_n)_{n \in \mathbb{N}}$ of partitions of $[0, T]$ tending to the identity.

The definition of LRM is very complicated to use. However, under Assumption 3.3.1, Theorem 1.6 of Schweizer [46] implies that the following definition of LRM is equivalent to original one:

Definition 3.3.5 *An L^2-strategy φ is said locally risk-minimizing for F if $V_T(\varphi) = 0$ and $C^F(\varphi)$ is a martingale orthogonal to M, that is, $C^F(\varphi)M$ is a martingale.*

Remark 3.3.6 *Note that φ is not self-financing. In fact, if φ is self-financing, then $C(\varphi)$ is a constant. If there exists a self-financing φ s.t. $V_T(\varphi) = 0$, we have $F = V_0(\varphi) + \int_0^T \xi_s dS_s$. This is a contradiction.*

We next define Föllmer-Schweizer decomposition (FS decomposition, for short).

Definition 3.3.7 *An $F \in L^2(\mathbb{P})$ admits a Föllmer-Schweizer decomposition if it can be described by*

$$F = F_0 + \int_0^T \xi_t^F dS_t + L_T^F, \tag{3.3.2}$$

where $F_0 \in \mathbb{R}$, $\xi^F \in \Theta_S$ and L^F is a square-integrable martingale orthogonal to M with $L_0^F = 0$.

Proposition 5.2 of Schweizer [46] shows the following:

Proposition 3.3.8 (Proposition 5.2 of Schweizer [46]) *Under Assumption 3.3.1, an LRM $\varphi = (\xi, \eta)$ for F exists if and only if F admits an FS decomposition, and its relationship is given by*

$$\xi_t = \xi_t^F, \quad \eta_t = F_0 + \int_0^t \xi_s^F dS_s + L_t^F - F1_{\{t=T\}} - \xi_t^F S_t.$$

We next define the minimal martingale measure.

Definition 3.3.9 (Minimal Martingale Measure) *A martingale measure $\mathbb{P}^*$, equivalent with the original measure $\mathbb{P}$, will be called minimal if $\mathbb{P}^* = \mathbb{P}$ on $\mathcal{F}$ and if any square-integrable $\mathbb{P}$-martingale which is orthogonal to the martingale part M of the semi-martingale X under $\mathbb{P}$ remains a martingale under $\mathbb{P}^*$.*

In the case S is continuous, we can get the FS decomposition by using the Galtchouk-Kunita-Watanabe decomposition under the minimal martingale measure.

Proposition 3.3.10 (Proposition of Vandaele and Vanmaele [52]) *If S is continuous, the locally risk-minimizing strategy is determined by the Galtchouk-Kunita-Watanabe decomposition under the minimal martingale measure.*

Unfortunately, in the case S is discontinuous, Vadaele and Vanmaele [52] showed that the locally risk-minimizing strategy is not determined by the Galtchouk-Kunita-Watanabe decomposition under the minimal martingale measure. Hence, there was no easy way to find the FS decomposition. In this thesis, we propose a useful way to find it by using the Malliavin calculus for canonical Lévy processes. To the end, in next chapter, we consider Malliavin calculus for Lévy processes and a Clark-Ocone type formula under change of measure for canonical Lévy processes.

Chapter 4

Malliavin calculus for Lévy processes and a Clark-Ocone type formula under change of measure for canonical Lévy processes

The Clark-Ocone formula is an explicit stochastic integral representation for random variables in terms of Malliavin derivatives. In this chapter, we prove a Clark-Ocone type formula under change of measure (COCM) for canonical Lévy processes with L^2-Lévy measure.

To show the COCM for L^2-Lévy processes, we develop Malliavin calculus for canonical Lévy processes, based on Solé et al. [49]. By using σ-finiteness of Lévy measure, we obtain a commutation formula for the Lebesgue integration and the Malliavin derivative and a chain rule for Malliavin derivative. These formulas derive the COCM. Finally, we obtain a log-Sobolev type formula for Lévy functionals.

The content of this chapter is based on Suzuki [50, 51].

4.1 Introduction

In this chapter, we develop Malliavin calculus for Lévy processes and derive a Clark-Ocone type formula under change of measure (COCM) for canonical Lévy processes.

The representations of functionals of Brownian motions (or Lévy processes) by stochastic integrals are important results in Probability theory. They have been widely studied (see, e.g., survey paper by Davis [16]). In particular, the Clark-Ocone (CO) formula is an explicit martingale representation of functionals of Brownian motions in terms of Malliavin derivatives. If an L^2-random variable F has certain regularity in the

Malliavin sense, we have

$$F = \mathbb{E}[F] + \int_0^T \mathbb{E}[D_t F|\mathcal{F}_t] dW_t$$

where W is a Brownian motion, $D_t F$ is the classical Malliavin derivative. This formula was shown by Clark, Ocone and Haussmann [13, 14, 23, 36]. A white noise version of the CO formula was proved by Aase et al. [1]. This formula has various applications. For example, the log-Sobolev and Poincare inequalities are obtained in Capitaine et al. [11]. In the application to mathematical finance, its representation of an optimal portfolio is given by this formula (see e.g., Ocone and Karatzas [35]).

The CO formula for Lévy processes has been also studied. Løkka [30] proved CO formula for functionals of pure jump Lévy processes. A white noise version of the CO formula for functionals of pure jump Lévy was derived by Di Nunno et al. [19]. Furthermore, we can also see that one for general L^2-Lévy functionals also holds (see Benth et al. [9]).

Since many applications in mathematical finance require representation of random variables with respect to risk neutral martingale measure, Girsanov transformations versions of this theorem were studied by many people. First, a Clark-Ocone type formula under change of measure (COCM) for Brownian motions was proved by Ocone and Karatzas [35]:

$$F = \mathbb{E}_{\mathbb{P}^*}[F] + \int_0^T \mathbb{E}_{\mathbb{P}^*}\left[D_t F - F \int_0^T D_t u_s dW_s^{\mathbb{P}^*} \middle| \mathcal{F}_t \right] dW_t^{\mathbb{P}^*}.$$

They also derived an optimal portfolio of Brownian market by using it. Okur [37] derive a white noise version of it and derived an explicit representation of hedging strategy of digital option for Brownian market. Huehne [24] derived a COCM for pure jump Lévy processes and gave an optimal portfolio. Note that Di Nunno et al. [20] and Okur [38] also introduced one for Lévy processes using white noise theory. However, their results are different from our results. Our results have different settings and different representation, for more detail, see Remark 4.5.6 and Theorem 4.5.3 in this chapter.

In this chapter, we derive a COCM for Lévy processes with L^2-Lévy measure in section 4.3:

$$\begin{aligned} F = &\mathbb{E}_{\mathbb{P}^*}[F] + \sigma \int_0^T \mathbb{E}_{\mathbb{P}^*}\left[D_{t,0}F - FK_t \middle| \mathcal{F}_{t-} \right] dW_t^{\mathbb{P}^*} \\ &+ \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[F(H_{t,z}^* - 1) + zH_{t,z}^* D_{t,z}F|\mathcal{F}_{t-}] \tilde{N}^{\mathbb{P}^*}(dt, dz). \end{aligned}$$

We precisely define K_t and $H_{t,z}^*$ and see sufficient conditions for this formula in section 4.5. Using this result, we obtain log-Sobolev and Poincare type inequalities for Lévy functionals. For that purpose, we adapted Malliavin calculus for Lévy processes based on Geiss and Laukkarinen [22] and Solé et al. [49]. Moreover, we show some formulas to show the main theorem, such as chain rule for Malliavin derivative and commutation formulas for integrals and the Malliavin derivative. By using σ-finiteness of Lévy measure (see e.g., Applebaum [3]), we prove it.

This chapter is organized as follows: In Section 4.2, we review Malliavin calculus for Lévy processes and we also give a chain rule. In Section 4.3, we first review commutation formulas like Delong and Imkeller [18] and we also review the Skorohod integral. Second, we give some comments about commutation formulas as a remark. Finally, we show another commutation formula. In Section 4.4, we review a Clark-Ocone type formula for canonical Lévy processes and Girsanov type theorem. In Section 4.5, by using results of Section 4.2, Section 4.3 and Section 4.4, we show a COCM for Lévy processes with L^2- Lévy measure. Using it, we obtain log-Sobolev and Poincare type inequalities for Lévy functionals.

4.2 Malliavin Calculus for canonical Lévy processes

4.2.1 Setting

We begin with preparation of the probabilistic framework and the underlying Lévy process X under which we discuss Malliavin calculus in the sequel. Let $T > 0$ be a finite time horizon, $(\Omega_W, \mathcal{F}_W, \mathbb{P}_W)$ a one-dimensional Wiener space on $[0, T]$; and W its coordinate mapping process, that is, a one-dimensional standard Brownian motion with $W_0 = 0$. Let $(\Omega_J, \mathcal{F}_J, \mathbb{P}_J)$ be the canonical Lévy space (see Solé et al. [49] and Delong and Imkeller [18]) for a pure jump Lévy process J on $[0, T]$ with Lévy measure ν, that is, $\Omega_J = \cup_{n=0}^{\infty}([0, T] \times \mathbb{R}_0)^n$, where $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$; and

$$J_t(\omega_J) = \sum_{i=1}^{n} z_i \mathbf{1}_{\{t_i \leq t\}}$$

for $t \in [0, T]$ and $\omega_J = ((t_1, z_1), \ldots, (t_n, z_n)) \in ([0, T] \times \mathbb{R}_0)^n$. Note that $([0, T] \times \mathbb{R}_0)^0$ represents an empty sequence. Now, we assume that $\int_{\mathbb{R}_0} z^2 \nu(dz) < \infty$; and denote $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega_W \times \Omega_J, \mathcal{F}_W \times \mathcal{F}_J, \mathbb{P}_W \times \mathbb{P}_J)$ and we call it canonical space. Let $\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0,T]}$ be the canonical filtration completed for $\mathbb{P}$. Let X be a square integrable centered Lévy process on $(\Omega, \mathcal{F}, \mathbb{P})$ represented as

$$X_t = \sigma W_t + J_t - t \int_{\mathbb{R}_0} z \nu(dz), \tag{4.2.1}$$

where $\sigma > 0$. Denoting by N the Poisson random measure defined as

$$N(t, A) := \sum_{s \leq t} \mathbf{1}_A(\Delta X_s),$$

$A \in \mathcal{B}(\mathbb{R}_0)$ and $t \in [0, T]$, where $\Delta X_s := X_s - X_{s-}$, we have $J_t = \int_0^t \int_{\mathbb{R}_0} z N(ds, dz)$. In addition, we define its compensated measure as $\widetilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt$. Thus, we can rewrite (4.2.1) as

$$X_t = \sigma W_t + \int_0^t \int_{\mathbb{R}_0} z \widetilde{N}(ds, dz). \tag{4.2.2}$$

We consider the finite measure q defined on $[0,T]\times\mathbb{R}$ by

$$q(E)=\sigma^2\int_{E(0)} dt\delta_0(dz)+\int_{E'} z^2 dt\nu(dz), \quad E\in\mathcal{B}([0,T]\times\mathbb{R}),$$

where $E(0)=\{(t,0)\in[0,T]\times\mathbb{R};(t,0)\in E\}$ and $E'=E-E(0)$, and the random measure Q on $[0,T]\times\mathbb{R}$ by

$$Q(E)=\sigma\int_{E(0)} dW_t\delta_0(dz)+\int_{E'} z\tilde{N}(dt,dz), \quad E\in\mathcal{B}([0,T]\times\mathbb{R}).$$

Let $L^2_{T,q,n}$ denote the set of product measurable, deterministic functions $h:([0,T]\times\mathbb{R})^n\to\mathbb{R}$ satisfying

$$\|h\|^2_{L^2_{T,q,n}}:=\int_{([0,T]\times\mathbb{R})^n}|h((t_1,z_1),\cdots,(t_n,z_n))|^2 q(dt_1,dz_1)\cdots q(dt_n,dz_n)<\infty.$$

For $n\in\mathbb{N}$ and $h_n\in L^2_{T,q,n}$, we denote

$$I_n(h_n):=\int_{([0,T]\times\mathbb{R})^n} h((t_1,z_1),\cdots,(t_n,z_n))Q(dt_1,dz_1)\cdots Q(dt_n,dz_n).$$

It is easy to see that $\mathbb{E}[I_0(h_0)]=h_0$ and $\mathbb{E}[I_n(h_n)]=0$, for $n\geq 1$. Moreover, this integral has the usual properties (see Itô [26]):

Proposition 4.2.1 *1. For $n\geq 1, f\in L^2_{T,q,n}$, we obtain, $I_n(f)=I_n(\tilde{f})$, where $\tilde{f}$ is the symmetrization of f :*

$$\tilde{f}((t_1,z_1),\cdots,(t_n,z_n))=\frac{1}{n!}\sum_{\pi\in\mathcal{D}_n} f((t_{\pi(1)},z_{\pi(1)}),\cdots,(t_{\pi(n)},z_{\pi(n)})),$$

where, $\mathcal{D}_n$ is the set of permutations of $\{1,2,\cdots,n\}$.

2. *For $n\geq 1, a,b\in\mathbb{R}, f,g\in L^2_{T,q,n}$, we get: $I_n(af+bg)=aI_n(f)+bI_n(g)$.*
3. *For $m,n\geq 1, f\in L^2_{T,q,n}, g\in L^2_{T,q,m}$, are symmetric in the n pairs $(t_i,z_i), 1\leq i\leq n$, that is $f=\tilde{f}$ and $g=\tilde{g}$, then, we have*

$$\mathbb{E}[I_n(f)I_m(g)]=n!\mathbf{1}_{(n=m)}\langle f,g\rangle_{L^2_{T,q,n}}.$$

In this setting, we introduce the following chaos expansion (see Theorem 2 in Itô[26], Section 2 of Solé[49] and Section 3 of Delong and Imkeller [18]).

Theorem 4.2.2 *Any $\mathcal{F}$-measurable square integrable random variable F on the canonical space has a unique representation*

$$F=\sum_{n=0}^{\infty} I_n(h_n), \mathbb{P}-\text{a.s.}$$

with functions $h_n \in L^2_{T,q,n}$ that are symmetric in the n pairs $(t_i, z_i), 1 \leq i \leq n$ and we have the isometry

$$\mathbb{E}[F^2] = \sum_{n=0}^{\infty} n! \|h_n\|^2_{L^2_{T,q,n}}.$$

By using the chaos expansion, we can define the following:

Definition 4.2.3 *(1) Let $\mathbb{D}^{1,2}$ denote the set of $\mathcal{F}$ -measurable random variables $F \in L^2(\mathbb{P})$ with the representation $F = \sum_{n=0}^{\infty} I_n(h_n)$ satisfying*

$$\sum_{n=1}^{\infty} nn! \|h_n\|^2_{L^2_{T,q,n}} < \infty.$$

(2) Let $F \in \mathbb{D}^{1,2}$. Then the Malliavin derivative $DF : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ of a random variable $F \in \mathbb{D}^{1,2}$ is a stochastic process defined by

$$D_{t,z}F := \sum_{n=1}^{\infty} nI_{n-1}(h_n((t,z),\cdot)), \quad \text{valid for } q\text{−a.e. } (t,z) \in [0,T] \times \mathbb{R}, \mathbb{P} - a.s.$$

(3) For $\sigma \neq 0$, let $\mathbb{D}^{1,2}_0$ denote the set of $\mathcal{F}$ -measurable random variables $F \in L^2(\mathbb{P})$ with the representation $F = \sum_{n=0}^{\infty} I_n(f_n)$ satisfying

$$\sum_{n=1}^{\infty} nn! \int_0^T \|f_n(\cdot,(t,0))\|^2_{L^2_{T,q,n-1}} \sigma^2 dt < \infty.$$

Then, for $F \in \mathbb{D}^{1,2}_0$, we can define

$$D_{t,0}F = \sum_{n=1}^{\infty} nI_{n-1}(f_n((t,0),\cdot)), \quad \text{valid for } q\text{−a.e. } (t,0) \in [0,T] \times \{0\}, \mathbb{P} - a.s.$$

(4) For $\nu \neq 0$, let $\mathbb{D}^{1,2}_1$ denote the set of $\mathcal{F}$ -measurable random variables $F \in L^2(\mathbb{P})$ with the representation $F = \sum_{n=0}^{\infty} I_n(f_n)$ satisfying

$$\sum_{n=1}^{\infty} nn! \int_0^T \int_{\mathbb{R}_0} \|f_n(\cdot,(t,z))\|^2_{L^2_{T,q,n-1}} z^2 \nu(dz) dt < \infty.$$

Then, for $F \in \mathbb{D}^{1,2}_1$, we can define

$$D_{t,z}F = \sum_{n=1}^{\infty} nI_{n-1}(f_n((t,z),\cdot)), \quad \text{valid for } q\text{−a.e. } (t,z) \in [0,T] \times \mathbb{R}_0, \mathbb{P} - a.s.$$

(5) Let D^W be the classical Malliavin derivative with respect to the Brownian motion W and Dom D^W *be the domain of D^W (for more details see Nualart [33] and Chapter 2). We define*

$$\mathbb{D}^W := \left\{ F \in L^2(\mathbb{P}); F(\cdot,\omega_N) \in \text{Dom } D^W \text{ for } \mathbb{P}^N\text{−a.e. } \omega_N \in \Omega_N \right\}.$$

(6) Let F be a random variable on $\Omega_W \times \Omega_N$. Then we define the increment quotient operator

$$\Psi_{t,z}F := \frac{F(\omega_W, \omega_N^{t,z}) - F(\omega_W, \omega_N)}{z}, z \neq 0,$$

where $\omega_N^{t,z}$ transforms a family $\omega_N = ((t_1, z_1), (t_2, z_2), \cdots) \in \Omega_N$ into a new family $\omega_N^{t,z} = ((t, z), (t_1, z_1), (t_2, z_2), \cdots) \in \Omega_N$, by adding a jump of size z at time t into the trajectory. Moreover, we denote

$$\mathbb{D}^J := \left\{ F \in L^2(\mathbb{P}); \mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} |\Psi_{t,z}F|^2 z^2 \nu(dz)dt\right] < \infty \right\}.$$

By Propositions 2.6.1, 2.6.2 in Delong [17] and result of Alós et al. [2] (see section 3.3), we can derive the following:

Proposition 4.2.4

1. *If $F \in \mathbb{D}^W$, then $F \in \mathbb{D}_0^{1,2}$ and $D_{t,0}F = \mathbf{1}_{\{\sigma>0\}}\sigma^{-1}D_t^W F(\cdot, \omega_N)(\omega_W)$ for q -a.e. $(t, z) \in [0, T] \times \{0\}, \mathbb{P}$ -a.s.*
2. *If $F \in \mathbb{D}^J$, then $F \in \mathbb{D}_1^{1,2}$ and $D_{t,z}F = \Psi_{t,z}F$ for q -a.e. $(t, z) \in [0, T] \times \mathbb{R}_0, \mathbb{P}$ -a.s.*
3. *$\mathbb{D}^{1,2} = \mathbb{D}^W \cap \mathbb{D}^J$ holds.*

Lemma 4.2.5 (Lemma 3.1 of Delong and Imkeller [18]) *Let $F \in \mathbb{D}^{1,2}$. Then, for $0 \leq t \leq T$, $\mathbb{E}[F|\mathcal{F}_t] \in \mathbb{D}^{1,2}$ and*

$$D_{s,x}\mathbb{E}[F|\mathcal{F}_t] = \mathbb{E}[D_{s,x}F|\mathcal{F}_t]\mathbf{1}_{\{s\leq t\}}, \text{ for } q\text{−a.e. } (s, x) \in [0, T] \times \mathbb{R}, \ \mathbb{P}\text{−a.s.}$$

We next establish the following fundamental result.

Proposition 4.2.6 (The closability of operator D) *Let $F \in L^2(\mathbb{P})$ and $F_k \in \mathbb{D}^{1,2}, k \in \mathbb{N}$ such that*

1. *$\lim_{k\to\infty} F_k = F$ in $L^2(\mathbb{P})$,*
2. *$\{D_{t,z}F_k\}_{k=1}^{\infty}$ converges in $L^2(q \times \mathbb{P})$.*

Then, $F \in \mathbb{D}^{1,2}$ and $\lim_{k\to\infty} D_{t,z}F_k = D_{t,z}F$ in $L^2(q \times \mathbb{P})$.

Proof. We can show this proposition by the same sort argument as Theorem 12.6 of Di Nunno et al. [20]. Let $F = \sum_{n=0}^{\infty} I_n(f_n), f_n \in L^2_{T,q,n}$ and $F_k = \sum_{n=0}^{\infty} I_n(f_n^k), f_n^k \in L^2_{T,q,n}$. Then by assumption (1), we have

$$\lim_{k\to\infty} \sum_{n=0}^{\infty} n!\|f_n^k - f_n\|^2_{L^2_{T,q,n}} = 0.$$

This implies that $\lim_{k\to\infty} f_n^k = f_n$ in $L^2_{T,q,n}$ for all n. From assumption (2), we deduce that

$$\lim_{k,m\to\infty}\sum_{n=1}^{\infty} nn!\|f_n^k - f_n^m\|^2_{L^2_{T,q,n}} = \lim_{k,m\to\infty}\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} (D_{t,z}F_k - D_{t,z}F_m)^2\, q(dt,dz)\right] = 0.$$

Hence, we obtain

$$\begin{aligned}\lim_{k\to\infty}\sum_{n=1}^{\infty} nn!\|f_n^k - f_n\|^2_{L^2_{T,q,n}} &\le 2\lim_{k\to\infty}\sum_{n=1}^{\infty}\liminf_{m\to\infty} nn!\|f_n^k - f_n^m\|^2_{L^2_{T,q,n}}\\ &\le 2\lim_{k\to\infty}\liminf_{m\to\infty}\sum_{n=1}^{\infty} nn!\|f_n^k - f_n^m\|^2_{L^2_{T,q,n}} = 0,\end{aligned}$$

because $nn!\|f_n^k - f_n^m\|^2_{L^2_{T,q,n}} \ge 0$ for all n, m, k.

Therefore, we can see that $F \in \mathbb{D}^{1,2}$ and $\lim_{k\to\infty} D_{t,z}F_k = D_{t,z}F$ in $L^2(q\times\mathbb{P})$. □

We next introduce a chain rule for the Malliavin derivatives. First we define the following.

Definition 4.2.7 1. *Let $C_0^\infty(\mathbb{R}^n)$ denote the space of smooth functions $f:\mathbb{R}^n\to\mathbb{R}$ with compact support.*

2. *A random variable of the form $F = f(X_{t_1},\cdots,X_{t_n})$, where $f\in C_0^\infty(\mathbb{R}^n)$, $n\in\mathbb{N}$, and $t_1,\cdots,t_n\ge 0$, is said to be a smooth random variable. The set of all smooth random variables is denoted by $\mathcal{S}$.*
3. *For $F\in\mathcal{S}$, we define the Malliavin derivative operator $\mathcal{D}$ as a map from $\mathcal{S}$ into $L^2(q\times\mathbb{P})$*

$$\begin{aligned}\mathcal{D}_{t,z}F := &\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(X_{t_1},\cdots,X_{t_n})\mathbf{1}_{[0,t_i]\times\{0\}}(t,z)\\ &+\frac{f(X_{t_1}+z\mathbf{1}_{[0,t_1]}(t),\cdots,X_{t_n}+z\mathbf{1}_{[0,t_n]}(t)) - f(X_{t_1},\cdots,X_{t_n})}{z}\mathbf{1}_{\mathbb{R}_0}(z)\end{aligned}$$

for $(t,z)\in[0,T]\times\mathbb{R}$.

By Lemma 3.1 and Theorem 4.1 in Geiss and Laukkarinen [22], we can see that the closure of the domain of $\mathcal{D}$ with respect to the norm

$$\|F\|_{\mathcal{D}} := \{\mathbb{E}[|F|^2] + \mathbb{E}[\|\mathcal{D}F\|^2_{L^2_q}]\}^{1/2}$$

is the space $\mathbb{D}^{1,2}$ and $D_{t,z}F = \mathcal{D}_{t,z}F$ for all $F\in S\subset\mathbb{D}^{1,2}$. Moreover, by Corollary 4.1 in Geiss and Laukkarinen [22], the set $\mathcal{S}$ of smooth random variables is dense in $L^2(\mathbb{P})$, $\mathbb{D}^{1,2}$, $\mathbb{D}^{1,2}_0$ and $\mathbb{D}^{1,2}_1$.

Proposition 4.2.8 *Let $\varphi:\mathbb{R}^n\to\mathbb{R}, n\ge 1$ be a C^1-function with bounded derivative.*

1. *If* $F = (F_1, \cdots, F_n) \in (\mathbb{D}_0^{1,2})^n$, *then,* $\varphi(F) \in \mathbb{D}_0^{1,2}$ *and*

$$D_{t,0}\varphi(F) = \sum_{k=1}^{n} \frac{\partial \varphi}{\partial x_k}(F) D_{t,0} F_k \mathbf{1}_{\{0\}}(z) \text{ for } q\text{−a.e. } (t,z) \in [0,T] \times \{0\},\ \mathbb{P}\text{−a.s.} \tag{4.2.3}$$

holds.

2. *If* $F = (F_1, \cdots, F_n) \in (\mathbb{D}_1^{1,2})^n$, *then* $\varphi(F) \in \mathbb{D}_1^{1,2}$ *and*

$$D_{t,z}\varphi(F) = \frac{\varphi(F_1 + zD_{t,z}F_1, \cdots, F_n + zD_{t,z}F_n) - \varphi(F_1, \cdots, F_n)}{z} \tag{4.2.4}$$

for q-a.e. $(t,z) \in [0,T] \times \mathbb{R}_0$, $\mathbb{P}$-a.s. *holds.*

Proof. (1) We can show this proposition by the same sort argument as Proposition 1.30 of Nualart [34]. We will only prove the case $n = 1$. The case $n > 1$ can be proved in the same way. Let $\varphi_m(x) = \int_{\mathbb{R}} \varphi(x-y)\psi_m(y)dy$, where, $\psi_m(x) = m\psi(mx)$, $m \in \mathbb{N}, x \in \mathbb{R}$, where, ψ is a C^∞ positive function with support $[-1,1]$ and $\int_{\mathbb{R}} \psi(x)dx = 1$. We can see that $\varphi_m \in C^\infty$ is bounded with bounded derivative. Since, $F \in \mathbb{D}_0^{1,2}$, there exists a sequence $\{F_k\}_{k=1}^\infty$, $F_k \in \mathcal{S}$, $F_k = f_k(X_{t_1}, \cdots, X_{t_{n_k}}), f_k \in C_0^\infty(\mathbb{R}^n)$ with $F_k \to F$ in $L^2(\mathbb{P})$ and $D_{t,0}F_k \to D_{t,0}F$ in $L^2(\lambda \times \mathbb{P})$. Then, we have

$$D_{t,0}\varphi_m(F_k) = \sum_{i=1}^{n_k} \partial_i(\varphi_m \circ f_k)(X_{t_1}, \cdots, X_{t_{n_k}}) = \varphi_m'(F_k)D_{t,0}F_k.$$

By using the triangle inequality,

$$\begin{aligned}
&\|\varphi_m'(F_k)D_{t,0}F_k - \varphi'(F)D_{t,0}F\|_{L^2(\lambda\times\mathbb{P})} \le \|\varphi_m'(F_k)(D_{t,0}F_k - D_{t,0}F)\|_{L^2(\lambda\times\mathbb{P})} \\
&+\|(\varphi_m'(F_k) - \varphi'(F_k))D_{t,0}F\|_{L^2(\lambda\times\mathbb{P})} + \|(\varphi'(F_k) - \varphi'(F))D_{t,0}F\|_{L^2(\lambda\times\mathbb{P})} \\
&=: I + II + III.
\end{aligned}$$

We can see that for any $m, k \ge 1$, $\varphi_m'(F_k)$ is bounded not depending on m and k, hence $I \to 0$ as $k \to \infty$. Moreover the dominated convergence theorem implies that for any $k \ge 1$, $II \to 0$ as $m \to \infty$. In the same way, we obtain $III \to 0$ as $k \to \infty$. Thus,

$$\lim_{k,m\to\infty} \|\varphi_m'(F_k)D_{t,0}F_k - \varphi'(F)D_{t,0}F\|_{L^2(\lambda\times\mathbb{P})} = 0.$$

Since $\lim_{m\to\infty} \varphi_m(x) = \varphi(x)$ uniformly and φ_m is a Lipschitz continuous function with Lipschitz constant not depending on m, we obtain $\lim_{k,m\to\infty} \varphi_m(F_k) = \varphi(F)$ in $L^2(\mathbb{P})$. Therefore, by the closability of $D_{t,0}$, we can see that $\varphi(F) \in \mathbb{D}_0^{1,2}$ and $D_{t,0}\varphi(F) = \varphi'(F)D_{t,0}F$.
(2) Equation (4.2.4) follows from the definition of the operator Ψ and Proposition 4.2.4.
□

Proposition 4.2.9 (Chain rule)
Let $\varphi \in C^1(\mathbb{R}^n;\mathbb{R})$ and $F = (F_1, \cdots, F_n)$, where $F_1, \cdots, F_n \in \mathbb{D}^{1,2}$. Suppose that $\varphi(F) \in L^2(\mathbb{P})$ and

$$\sum_{k=1}^n \frac{\partial}{\partial x_k}\varphi(F) D_{t,0}F_k \mathbf{1}_{\{0\}}(z) \\ + \frac{\varphi(F_1 + zD_{t,z}F_1, \cdots, F_n + zD_{t,z}F_n) - \varphi(F_1, \cdots, F_n)}{z}\mathbf{1}_{\mathbb{R}_0}(z) \in L^2(q \times \mathbb{P}).$$

Then, we obtain $\varphi(F) \in \mathbb{D}^{1,2}$ and

$$D_{t,z}\varphi(F) = \sum_{k=1}^n \frac{\partial \varphi}{\partial x_k}(F) D_{t,0}F_k \mathbf{1}_{\{0\}}(z) \\ + \frac{\varphi(F_1 + zD_{t,z}F_1, \cdots, F_n + zD_{t,z}F_n) - \varphi(F_1, \cdots, F_n)}{z}\mathbf{1}_{\mathbb{R}_0}(z).$$

Proof. We can show this proposition by the same sort argument as Lemma A.1 of Ocone-Karatzas [35]. Let $\Psi \in C_0^\infty(\mathbb{R})$ satisfy $\Psi(y) = y$ if $|y| \le 1$, $|\Psi(y)| \le |y|$ for all $y \in \mathbb{R}$. For any $l \in \mathbb{N}$, let $\varphi_l(x) = l\Psi(\frac{\varphi(x)}{l})$, $x \in \mathbb{R}^n$. For each l, $\varphi_l \in C_b^1(\mathbb{R}^n;\mathbb{R})$ and thus $\varphi_l(F) \in \mathbb{D}^{1,2}$ and

$$D_{t,z}\varphi_l(F) = \Psi'(\varphi(F)/l) \sum_{k=1}^n \frac{\partial \varphi}{\partial x_k}(F) D_{t,0}F_k \mathbf{1}_{\{0\}}(z) \\ + \frac{\varphi_l(F_1 + zD_{t,z}F_1, \cdots, F_n + zD_{t,z}F_n) - \varphi_l(F_1, \cdots, F_n)}{z}\mathbf{1}_{\mathbb{R}_0}(z)$$

by Proposition 4.2.8. Note that $|\varphi_l(F)| \le |\varphi(F)|$ for all l, $\lim_{l\to\infty} \varphi_l(F) = \varphi(F)$ a.s. and

$$\begin{aligned}\lim_{l\to\infty} D_{t,z}\varphi_l(F) &= \sum_{k=1}^n \frac{\partial \varphi}{\partial x_k}(F) D_{t,0}F_k \mathbf{1}_{\{0\}}(z) \\ &\quad + \frac{\varphi(F_1 + zD_{t,z}F_1, \cdots, F_n + zD_{t,z}F_n) - \varphi(F_1, \cdots, F_n)}{z}\mathbf{1}_{\mathbb{R}_0}(z) \\ &=: I_\infty\end{aligned}$$

$q \times \mathbb{P}$-a.e. Moreover note that

$$\begin{aligned}&|D_{t,z}\varphi_l(F) - I_\infty| \\ &\le |\Psi'(\varphi(F)/l) \sum_{k=1}^n \frac{\partial \varphi}{\partial x_k}(F) D_{t,0}F_k \mathbf{1}_{\{0\}}(z) - \sum_{k=1}^n \frac{\partial \varphi}{\partial x_k}(F) D_{t,0}F_k \mathbf{1}_{\{0\}}(z)| \\ &\quad + |\frac{\varphi_l(F_1 + zD_{t,z}F_1, \cdots, F_n + zD_{t,z}F_n) - \varphi_l(F_1, \cdots, F_n)}{z}\mathbf{1}_{\mathbb{R}_0}(z) \\ &\quad - \frac{\varphi(F_1 + zD_{t,z}F_1, \cdots, F_n + zD_{t,z}F_n) - \varphi(F_1, \cdots, F_n)}{z}\mathbf{1}_{\mathbb{R}_0}(z)| \\ &\le (\sup_{y\in\mathbb{R}} |\Psi'(y)| + 1)| \sum_{k=1}^n \frac{\partial \varphi}{\partial x_k}(F) D_{t,0}F_k |\mathbf{1}_{\{0\}}(z)\end{aligned}$$

$$+\sup_{y\in\mathbb{R}}|\Psi'(y)|\sqrt{\sum_{k=1}^{n}(D_{t,z}F_k)^2}\mathbf{1}_{\mathbb{R}_0}(z)$$
$$+\left|\frac{\varphi(F_1+zD_{t,z}F_1,\cdots,F_n+zD_{t,z}F_n)-\varphi(F_1,\cdots,F_n)}{z}\right|\mathbf{1}_{\mathbb{R}_0}(z)\in L^2(q\times\mathbb{P})$$

Therefore dominated convergence theorem implies that $\lim_{l\to\infty}\varphi_l(F)=\varphi(F)$ in $L^2(\mathbb{P})$ and $\lim_{l\to\infty}D_{t,z}\varphi_l(F)=I_\infty$ in $L^2(q\times\mathbb{P})$. Hence, Proposition 4.2.6 implies that $\varphi(F)\in\mathbb{D}^{1,2}$ and

$$D_{t,z}\varphi(F)=\sum_{k=1}^{n}\frac{\partial\varphi}{\partial x_k}(F)D_{t,0}F_k\mathbf{1}_{\{0\}}(z)$$
$$+\frac{\varphi(F_1+zD_{t,z}F_1,\cdots,F_n+zD_{t,z}F_n)-\varphi(F_1,\cdots,F_n)}{z}\mathbf{1}_{\mathbb{R}_0}(z).$$

□

If we take $\varphi(x,y)=xy$, then, we can derive the following product rule.

Corollary 4.2.10 *Let $F_1,F_2\in\mathbb{D}^{1,2}$ and $F_1F_2\in L^2(\mathbb{P})$. Moreover, assume that $F_1D_{t,z}F_2+F_2D_{t,z}F_1+zD_{t,z}F_1\cdot D_{t,z}F_2\in L^2(q\times\mathbb{P})$. Then $F_1F_2\in\mathbb{D}^{1,2}$ and*

$$D_{t,z}F_1F_2=F_1D_{t,z}F_2+F_2D_{t,z}F_1+zD_{t,z}F_1\cdot D_{t,z}F_2\quad q-\text{a.e. }(t,z)\in[0,T]\times\mathbb{R},\mathbb{P}-a.s. \tag{4.2.5}$$

4.3 The Skorohod integral and commutation of integration and the Malliavin differentiability

In this section, we consider the Skorohod integral and commutation of integration and the Malliavin differentiability, which has an interest of its own and could be applied for other purposes than the one of this chapter. First we introduce the following classes.

Definition 4.3.1 *(1) Let $\mathbb{L}^{1,2}$ denote the space of product measurable and $\mathbb{F}$ -adapted processes $G:\Omega\times[0,T]\times\mathbb{R}\to\mathbb{R}$ satisfying*

$$\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}}|G_{s,x}|^2q(ds,dx)\right]<\infty,$$

$G_{s,x}\in\mathbb{D}^{1,2},q-$a.e. $(s,x)\in[0,T]\times\mathbb{R}$ *and*

$$\mathbb{E}\left[\int_{([0,T]\times\mathbb{R})^2}|D_{t,z}G_{s,x}|^2q(ds,dx)q(dt,dz)\right]<\infty.$$

(2) $\mathbb{L}_0^{1,2}$ denotes the space of $G:[0,T]\times\Omega\to\mathbb{R}$ satisfying

1. $G_s \in \mathbb{D}^{1,2}$ *for a.e.* $s \in [0,T]$,
2. $E\left[\int_{[0,T]} |G_s|^2 ds\right] < \infty$,
3. $E\left[\int_{[0,T]\times\mathbb{R}} \int_0^T |D_{t,z}G_s|^2 ds q(dt,dz)\right] < \infty$.

(3) $\mathbb{L}_1^{1,2}$ *is defined as the space of* $G : [0,T] \times \mathbb{R}_0 \times \Omega \to \mathbb{R}$ *such that*

1. $G_{s,x} \in \mathbb{D}^{1,2}$ *for q-a.e.* $(s,x) \in [0,T] \times \mathbb{R}$,
2. $E\left[\int_{[0,T]\times\mathbb{R}_0} |G_{s,x}|^2 \nu(dx) ds\right] < \infty$,
3. $E\left[\int_{[0,T]\times\mathbb{R}} \int_{[0,T]\times\mathbb{R}_0} |D_{t,z}G_{s,x}|^2 \nu(dx) ds q(dt,dz)\right] < \infty$.

(4) $\tilde{\mathbb{L}}_1^{1,2}$ *is defined as the space of* $G \in \mathbb{L}^{1,2}$ *such that*

1. $E\left[\left(\int_{[0,T]\times\mathbb{R}_0} |G_{s,x}| \nu(dx) ds\right)^2\right] < \infty$,
2. $E\left[\int_{[0,T]\times\mathbb{R}} \left(\int_{[0,T]\times\mathbb{R}_0} |D_{t,z}G_{s,x}| \nu(dx) ds\right)^2 q(dt,dz)\right] < \infty$.

(5) Recall that any function $u \in L^2(q \times \mathbb{P})$ *has a chaotic representation*

$$u_{t,z} = \sum_{n=0}^{\infty} I_n(h_n(\cdot,(t,z))),$$

where $h_n \in L^2_{T,q,n+1}$ *is symmetric in the first n pairs of variables. Denoting by* $\hat{h}_n$ *the symmetrization of* h_n *with respect to all* $n+1$ *pairs of variables, we define*

$$\mathrm{Dom}_\delta := \left\{ u \in L^2(q \times \mathbb{P}) \Big| \sum_{n=0}^{\infty} (n+1)! \|\hat{h}_n\|^2_{L^2_{T,q,n+1}} < \infty \right\}.$$

(6) Let $u \in \mathrm{Dom}_\delta$*. Then the Skorohod integral* δ *with respect to the random measure Q of a process* $u : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ *is defined as*

$$\delta(u) = \sum_{n=0} I_{n+1}(\hat{h}_n),\ \mathbb{P}\text{−a.s.}$$

The Skorohod integral δ has the following properties (see section 6 of Solé et al. [49]):

Proposition 4.3.2 (1) Duality formula

A process $u \in L^2(q \times \mathbb{P})$ *belongs to* Dom_δ *if and only if there exists a constant C such that for all* $F \in \mathbb{D}^{1,2}$,

$$\left| \mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} u_{s,x} D_{s,x} F q(ds,dx)\right] \right| \leq C(\mathbb{E}[F^2])^{1/2}.$$

If $u \in \mathrm{Dom}_\delta$*, then* $\delta(u)$ *is the element of* $L^2(\mathbb{P})$ *characterized by*

$$\mathbb{E}[\delta(u)F] = \mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} u(s,x) D_{s,x} F q(ds,dx)\right]$$

for any $F \in \mathbb{D}^{1,2}$.

(2) Covariance of Skolohod integrals

A process $u \in L^2(q \times \mathbb{P})$ *belongs to* $\mathbb{L}^{1,2}$ *if and only if*

$$\sum_{n=1}^{\infty} n \cdot n! \|\hat{h}_n\|^2_{L^2_{T,q,n+1}} < \infty$$

holds, and, in particular, this implies $\mathbb{L}^{1,2} \subset \mathrm{Dom}_\delta$. *For* $u, v \in \mathbb{L}^{1,2}$,

$$\begin{aligned}\mathbb{E}[\delta(u)\delta(v)] &= \mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} u(s,x)v(s,x)q(ds,dx)\right] \\ &\quad + \mathbb{E}\left[\int_{([0,T]\times\mathbb{R})^2} D_{t,z}u(s,x)D_{t,z}v(s,x)q(dt,dz)q(ds,dx)\right].\end{aligned}$$

(3) Differentiability of δ

Let $u \in \mathbb{L}^{1,2}$ *such that* $D_{t,z}u \in \mathrm{Dom}_\delta$ *for all* $(t,z) \in [0,T] \times \mathbb{R}$, *q-a.e. Then* $\delta(u) \in \mathbb{D}^{1,2}$ *and*

$$D_{t,z}\delta(u) = u_{t,z} + \delta(D_{t,z}u)$$

for all $(t,z) \in [0,T] \times \mathbb{R}$, *q-a.e.*

(4) Skorohod integral is an extension of the Itô integral

Let $u \in L^2(q \times \mathbb{P})$ *be predictable. Then,* $u \in \mathrm{Dom}_\delta$ *and*

$$\delta(u) = \int_{[0,T]\times\mathbb{R}} u_{s,x}Q(ds,dx).$$

We next discuss the commutation relation of the stochastic integral with the Malliavin derivative.

Proposition 4.3.3 (Lemma 3.3 of Delong and Imkeller [18])
Let $G : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ *be a predictable process with*

$$\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} |G_{s,x}|^2 q(ds,dx)\right] < \infty.$$

Then

$$G \in \mathbb{L}^{1,2} \text{ if and only if } \int_{[0,T]\times\mathbb{R}} G_{s,x}Q(ds,dx) \in \mathbb{D}^{1,2}.$$

Furthermore, if $\int_{[0,T]\times\mathbb{R}} G_{s,x}Q(ds,dx) \in \mathbb{D}^{1,2}$, *then, for q -a.e.* $(t,z) \in [0,T] \times \mathbb{R}$, *we have*

$$D_{t,z}\int_{[0,T]\times\mathbb{R}} G_{s,x}Q(ds,dx) = G_{t,z} + \int_{[0,T]\times\mathbb{R}} D_{t,z}G_{s,x}Q(ds,dx), \quad \mathbb{P}-\text{a.s.},$$

and $\int_{[0,T]\times\mathbb{R}} D_{t,z}G_{s,x}Q(ds,dx)$ *is a stochastic integral in Itô sense.*

Next proposition provides commutation of the Lebesgue integration and the Malliavin differentiability.

Proposition 4.3.4 (Lemma 3.2 of Delong and Imkeller [18])
Assume that $G : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ is a product measurable and $\mathbb{F}$ -adapted process, η on $[0,T] \times \mathbb{R}$ a finite measure, so that conditions

$$\begin{aligned} &\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} |G_{s,x}|^2 \eta(ds,dx)\right] < \infty, \\ &G_{s,x} \in \mathbb{D}^{1,2}, \quad \text{for } \eta\text{-a.e. } (s,x) \in [0,T]\times\mathbb{R}, \\ &\mathbb{E}\left[\int_{([0,T]\times\mathbb{R})^2} |D_{t,z}G_{s,x}|^2 \eta(ds,dx) q(dt,dz)\right] < \infty \end{aligned}$$

are satisfied. Then we have

$$\int_{[0,T]\times\mathbb{R}} G_{s,x}\eta(ds,dx) \in \mathbb{D}^{1,2}$$

and the differentiation rule

$$D_{t,z}\int_{[0,T]\times\mathbb{R}} G_{s,x}\eta(ds,dx) = \int_{[0,T]\times\mathbb{R}} D_{t,z}G_{s,x}\eta(ds,dx)$$

holds for q -a.e. $(t,z) \in [0,T]\times\mathbb{R}, \mathbb{P}$ -a.s.

Remark 4.3.5 *We already know the following:*

1. *If $G(s,x) \in L^1(\eta)$ is a deterministic function, and $\eta([0,T]\times\mathbb{R}) < \infty$ or $\eta([0,T]\times\mathbb{R}) = \infty$, then we can see $\int_{[0,T]\times\mathbb{R}} G(s,x)\eta(ds,dx) \in \mathbb{D}^{1,2}$ and $D_{t,z}\int_{[0,T]\times\mathbb{R}} G(s,x)\eta(ds,dx) = 0 = \int_{[0,T]\times\mathbb{R}} D_{t,z}G(s,x)\eta(ds,dx)$.*
2. *Let $\eta(dx,ds) = \delta_{\mathbb{R}_0}(x)\nu(dx)ds$ with $\nu(\mathbb{R}_0) < \infty$. Then, Proposition 4.3.4 implies that $\int_{[0,T]\times\mathbb{R}_0} G(s,x)\nu(dx)ds \in \mathbb{D}^{1,2}$ and the differentiation rule holds.*
3. *We assume ν satisfies $\nu(\mathbb{R}_0) < \infty$ or $\nu(\mathbb{R}_0) = \infty$. Moreover if $G(s,x) = g_1(x)g_2(s)$, where, $g_1(x) \in L^1(\nu)$ is a deterministic function and $g_2(s) \in \mathbb{L}_0^{1,2}$ is a stochastic process, then, we have $\int_{[0,T]\times\mathbb{R}_0} G(s,x)\nu(dx)ds = \int_{\mathbb{R}_0} g_1(x)\nu(dx)\int_{[0,T]} g_2(s)ds = C\int_{[0,T]} g_2(s)ds$, where $C := \int_{\mathbb{R}_0} g_1(x)\nu(dx)$ is a constant number. Therefore, by Proposition 4.3.4, we can see $C\int_{[0,T]} g_2(s)ds \in \mathbb{D}^{1,2}$ and the differentiation rule holds.*

By using σ-finiteness of ν and Proposition 4.3.4, we can show the following proposition.

Proposition 4.3.6 *Let $G \in \tilde{\mathbb{L}}_1^{1,2}$. Then,*

$$\int_{[0,T]\times\mathbb{R}_0} G_{s,x}\nu(dx)ds \in \mathbb{D}^{1,2}$$

and the differentiation rule

$$D_{t,z}\int_{[0,T]\times\mathbb{R}_0} G_{s,x}\nu(dx)ds = \int_{[0,T]\times\mathbb{R}_0} D_{t,z}G_{s,x}\nu(dx)ds$$

holds for q -a.e. $(t,z) \in [0,T]\times\mathbb{R}, \mathbb{P}$ -a.s.

Proof. Since ν is σ-finite measure, we can find a sequence $(A_n, n \in \mathbb{N})$ in $\mathcal{B}(\mathbb{R}_0)$ such that $\mathbb{R}_0 = \bigcup_{n=1}^{\infty} A_n$ and $\nu(A_n) < \infty$. Hence, Proposition 4.3.4 implies

$$\int_{[0,T]\times\bigcup_{n=1}^{k} A_n} G(s,x)\nu(dx)ds \in \mathbb{D}^{1,2}, k \in \mathbb{N}$$

and

$$D_{t,z}\int_{[0,T]\times\bigcup_{n=1}^{k} A_n} G(s,x)\nu(dx)ds = \int_{[0,T]\times\bigcup_{n=1}^{k} A_n} D_{t,z}G(s,x)\nu(dx)ds.$$

Next, note the following;

$$\lim_{k\to\infty} G(s,x)\mathbf{1}_{\bigcup_{n=1}^{k} A_n}(x) = G(s,x), \nu\otimes\lambda\otimes\mathbb{P}-\text{a.e.},$$

hence,

$$\lim_{k\to\infty} G(s,x)\mathbf{1}_{\bigcap_{n=1}^{k} A_n^C}(x) = 0, \nu\otimes\lambda\otimes\mathbb{P}-\text{a.e.},$$

$$|G(s,x)\mathbf{1}_{\bigcup_{n=1}^{k} A_n}(x) - G(s,x)| = |G(s,x)\mathbf{1}_{\bigcap_{n=1}^{k} A_n^C}(x)| \le |G(s,x)| \in L^1(\nu\times\lambda)$$

and

$$\begin{aligned}&\left|\int_{[0,T]\times\mathbb{R}_0} G(s,x)\nu(dx)ds - \int_{[0,T]\times\bigcup_{n=1}^{k} A_n} G(s,x)\nu(dx)ds\right|^2\\ &\le \left(\int_{[0,T]\times\mathbb{R}_0} |G(s,x)|\nu(dx)ds\right)^2 \in L^1(\mathbb{P}).\end{aligned}$$

Then, by Lebesgue's dominated convergence theorem, we can see

$$\lim_{k\to\infty}\mathbb{E}\left[\left|\int_{[0,T]\times\mathbb{R}_0} G(s,x)\nu(dx)ds - \int_{[0,T]\times\bigcup_{n=1}^{k} A_n} G(s,x)\nu(dx)ds\right|^2\right] = 0.$$

Moreover,

$$\lim_{k\to\infty} D_{t,z}G(s,x)\mathbf{1}_{\bigcup_{n=1}^{k} A_n}(x) = D_{t,z}G(s,x), \nu\otimes\lambda\otimes\mathbb{P}\otimes q-\text{a.e.},$$

hence,

$$\lim_{k\to\infty} D_{t,z}G(s,x)\mathbf{1}_{\bigcap_{n=1}^{k} A_n^C}(x) = 0, \nu\otimes\lambda\otimes\mathbb{P}\otimes q-\text{a.e.},$$

$$\begin{aligned}&|D_{t,z}G(s,x)\mathbf{1}_{\bigcup_{n=1}^{k} A_n}(x) - D_{t,z}G(s,x)| = |D_{t,z}G(s,x)\mathbf{1}_{\bigcap_{n=1}^{k} A_n^C}(x)|\\ &\le |D_{t,z}G(s,x)| \in L^1(\nu\times\lambda),\end{aligned}$$

and

$$\left|\int_{[0,T]\times\mathbb{R}_0} D_{t,z}G(s,x)\nu(dx)ds - \int_{[0,T]\times\bigcup_{n=1}^k A_n} D_{t,z}G(s,x)\nu(dx)ds\right|^2$$
$$\leq \left(\int_{[0,T]\times\mathbb{R}_0} |D_{t,z}G(s,x)|\nu(dx)ds\right)^2 \in L^1(q\times\mathbb{P}).$$

Then, Lebesgue's dominated convergence theorem shows

$$\int_{[0,T]\times\mathbb{R}} \mathbb{E}\left[\left|\int_{[0,T]\times\mathbb{R}_0} D_{t,z}G(s,x)\nu(dx)ds - \int_{[0,T]\times\,\cup_{n=1}^k A_n} D_{t,z}G(s,x)\nu(dx)ds\right|^2\right]$$

$\times q(dt,dz) \to 0$ as $k\to\infty$. Therefore, by Proposition 4.2.6, we can conclude

$$\int_{[0,T]\times\mathbb{R}_0} G(s,x)\nu(dx)ds \in \mathbb{D}^{1,2}$$

and the differentiation rule

$$D_{t,z}\int_{[0,T]\times\mathbb{R}_0} G(s,x)\nu(dx)ds = \int_{[0,T]\times\mathbb{R}_0} D_{t,z}G(s,x)\nu(dx)ds$$

holds for q -a.e. $(t,z)\in[0,T]\times\mathbb{R},\mathbb{P}$ -a.s. □

4.4 Clark-Ocone type formula for canonical Lévy functionals and Girsanov type theorem

4.4.1 Clark-Ocone type formula for canonical Lévy functionals

We next present an explicit form of the martingale representation formula by using Malliavin calculus (see e.g., Theorem 3.5.2 in Delong [17]).

Proposition 4.4.1 (Clark-Ocone type formula for canonical Lévy functionals)
Let $F\in\mathbb{D}^{1,2}$. Then, we have

$$\begin{aligned} F &= \mathbb{E}[F] + \int_{[0,T]\times\mathbb{R}} \mathbb{E}[D_{t,z}F|\mathcal{F}_{t-}]Q(dt,dz) \\ &= \mathbb{E}[F] + \sigma\int_0^T \mathbb{E}[D_{t,0}F|\mathcal{F}_{t-}]dW_t + \int_0^T\int_{\mathbb{R}_0} \mathbb{E}[D_{t,z}F|\mathcal{F}_{t-}]z\tilde{N}(dt,dz). \end{aligned} \tag{4.4.6}$$

Proof. We introduce two proofs.
(1) First proof is equal to the one for the Brownian motion case (see, Theorem 4.1 in Di

Nunno et al. [20]) and pure jump Lévy case (see, Theorem 12.16 in Di Nunno et al. [20]). We denote $F = \sum_{n=0}^{\infty} I_n(f_n), f_n \in L^2_{T,q,n}$ and

$$J_n(f_n) := \int_0^T \int_{\mathbb{R}} \cdots \int_0^{t_2-} \int_{\mathbb{R}} f_n(t_1, z_1, \cdots, t_n, z_n) Q(dt_1, dz_1) \cdots Q(dt_n, dz_n) = \frac{1}{n!} I_n(f_n)$$

where $0 \le t_1 \le \cdots t_n \le T$. From $\mathbb{E}[F] = I_0(f_0), D_{t,z}F = \sum_{n=1}^{\infty} n I_{n-1}(f_n((t,z),\cdot))$, We obtain

$$\begin{aligned}
&\mathbb{E}[F] + \int_0^T \int_{\mathbb{R}} \mathbb{E}[D_{t,z}F | \mathcal{F}_{t-}] Q(dt, dz) \\
&= I_0(f_0) + \int_0^T \int_{\mathbb{R}} \mathbb{E}[\sum_{n=1}^{\infty} n I_{n-1}(f_n((t,z),\cdot)) | \mathcal{F}_{t-}] Q(dt, dz) \\
&= I_0(f_0) + \int_0^T \int_{\mathbb{R}} \mathbb{E}[\sum_{n=1}^{\infty} n(n-1)! J_{n-1}(f_n((t,z),\cdot)) | \mathcal{F}_{t-}] Q(dt, dz) \\
&= I_0(f_0) + \sum_{n=1}^{\infty} n! \int_0^T \int_{\mathbb{R}} \mathbb{E}[\int_0^T \int_{\mathbb{R}} \cdots \int_0^{t_2-} \int_{\mathbb{R}} f_n(t_1, z_1, \cdots, t_{n-1}, z_{n-1}, t, z) \\
&\quad \times Q(dt_1, dz_1) \cdots Q(dt_{n-1}, dz_{n-1}) | \mathcal{F}_{t-}] Q(dt, dz) \\
&= I_0(f_0) + \sum_{n=1}^{\infty} n! \int_0^T \int_{\mathbb{R}} \int_0^{t-} \int_{\mathbb{R}} \cdots \int_0^{t_2-} \int_{\mathbb{R}} f_n(t_1, z_1, \cdots, t_{n-1}, z_{n-1}, t, z) \\
&\quad \times Q(dt_1, dz_1) \cdots Q(dt_{n-1}, dz_{n-1}) Q(dt, dz) \\
&= I_0(f_0) + \sum_{n=1}^{\infty} n! \int_0^T \int_{\mathbb{R}} \int_0^{t_n-} \int_{\mathbb{R}} \cdots \int_0^{t_2-} \int_{\mathbb{R}} f_n(t_1, z_1, \cdots, t_{n-1}, z_{n-1}, t_n, z_n) \\
&\quad \times Q(dt_1, dz_1) \cdots Q(dt_{n-1}, dz_{n-1}) Q(dt_n, dz_n) \\
&= I_0(f_0) + \sum_{n=1}^{\infty} n! J_n(f_n) \\
&= I_0(f_0) + \sum_{n=1}^{\infty} I_n(f_n) = \sum_{n=0}^{\infty} I_n(f_n) = F.
\end{aligned}$$

(2) The martingale representation theorem (see, e.g. Proposition 9.4 of Cont and Tankov [15]) provides that

$$\begin{aligned}
F &= \mathbb{E}[F] + \int_0^T \varphi^{(1)}_{s-} dW_s + \int_0^T \int_{\mathbb{R}_0} \varphi^{(2)}_{s-,x} \tilde{N}(ds, dx) \\
&= \mathbb{E}[F] + \int_0^T \frac{\varphi^{(1)}_{s-}}{\sigma} \sigma dW_s + \int_0^T \int_{\mathbb{R}_0} \frac{\varphi^{(2)}_{s-,x}}{x} x \tilde{N}(ds, dx) \\
&= \mathbb{E}[F] + \int_0^T \int_{\mathbb{R}} \left(\frac{\varphi^{(1)}_{s-}}{\sigma} \mathbf{1}_{\{0\}}(x) + \frac{\varphi^{(2)}_{s-,x}}{x} \mathbf{1}_{\mathbb{R}_0}(x) \right) Q(ds, dx),
\end{aligned}$$

where $\varphi^{(1)}$ and $\varphi^{(2)}$ are $L^2(q \times \mathbb{P})$-predictable processes. Since $F \in \mathbb{D}^{1,2}$, Proposition 4.3.4 implies that

$$D_{t,z}F = \frac{\varphi^{(1)}_{t-}}{\sigma}\mathbf{1}_{\{0\}}(z) + \frac{\varphi^{(2)}_{t-,z}}{z}\mathbf{1}_{\mathbb{R}_0}(z) + \int_{t-}^{T}\int_{\mathbb{R}} D_{t,z}\left(\frac{\varphi^{(1)}_{s-}}{\sigma}\mathbf{1}_{\{0\}}(x) + \frac{\varphi^{(2)}_{s-,x}}{x}\mathbf{1}_{\mathbb{R}_0}(x)\right) Q(ds,dx).$$

Hence, we have

$$\mathbb{E}[D_{t,z}F|\mathcal{F}_{t-}] = \frac{\varphi^{(1)}_{t-}}{\sigma}\mathbf{1}_{\{0\}}(z) + \frac{\varphi^{(2)}_{t-,z}}{z}\mathbf{1}_{\mathbb{R}_0}(z).$$

Therefore, we can see that $\varphi^{(1)}_{t-} = \sigma\mathbb{E}[D_{t,0}F|\mathcal{F}_{t-}]$ and $\varphi^{(2)}_{t-,z} = z\mathbb{E}[D_{t,z}F|\mathcal{F}_{t-}]$. □

4.4.2 Girsanov theorem for Lévy processes

We recall the Girsanov theorem for Lévy processes (see, e.g., Theorem 12.21 of Di Nunno et al. [20]).

Theorem 4.4.2 *Let $\theta_{s,x} < 1, s \in [0,T], x \in \mathbb{R}_0$ and $u_s, s \in [0,T]$, be predictable processes such that*

$$\int_0^T \int_{\mathbb{R}_0} \{|\log(1-\theta_{s,x})|^2 + \theta_{s,x}^2\}\nu(dx)ds < \infty, \text{ a.s.,}$$
$$\int_0^T u_s^2 ds < \infty, \text{ a.s.}$$

Moreover we denote

$$Z_t := \exp\left(-\int_0^t u_s dW_s - \frac{1}{2}\int_0^t u_s^2 ds + \int_0^t \int_{\mathbb{R}_0} \log(1-\theta_{s,x})\tilde{N}(ds,dx) + \int_0^t \int_{\mathbb{R}_0} (\log(1-\theta_{s,x}) + \theta_{s,x})\nu(dx)ds\right), t \in [0,T].$$

Define a measure $\mathbb{P}^*$ on $\mathcal{F}_T$ by

$$d\mathbb{P}^*(\omega) = Z_T(\omega)d\mathbb{P}(\omega),$$

and we assume that $Z(T)$ satisfies the Novikov condition, that is,

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T u_s^2 ds + \int_0^T \int_{\mathbb{R}_0} \{(1-\theta_{s,x})\log(1-\theta_{s,x}) + \theta_{s,x}\}\nu(dx)ds\right)\right] < \infty.$$

Then $\mathbb{E}[Z_T] = 1$ and hence $\mathbb{P}^*$ is a probability measure on $\mathcal{F}_T$. Furthermore if we denote

$$\tilde{N}^{\mathbb{P}^*}(dt,dx) := \theta_{t,z}\nu(dx)dt + \tilde{N}(dt,dx)$$

and

$$dW_t^{\mathbb{P}^*} := u_t dt + dW_t,$$

then $\tilde{N}^{\mathbb{P}^*}(\cdot,\cdot)$ and $W^{\mathbb{P}^*}(\cdot)$ are the compensated Poisson random measure of $N(\cdot,\cdot)$ and a standard Brownian motion under $\mathbb{P}^*$, respectively.

4.5 A Clark-Ocone type formula under change of measure for canonical Lévy processes

4.5.1 A Clark-Ocone type formula under change of measure for canonical Lévy processes

In this section, we introduce a Clark-Ocone type formula under change of measure for canonical Lévy processes. Throughout this section, under the same setting as Theorem 4.4.2, we assume the following.

Assumption 4.5.1
(1) $u, u^2 \in \mathbb{L}_0^{1,2}$; and $2u_s D_{t,z} u_s + z(D_{t,z}u_s)^2 \in L^2(q \times \mathbb{P})$ for a.e. $s \in [0,T]$.
(2) $\theta + \log(1-\theta) \in \tilde{\mathbb{L}}_1^{1,2}$, and $\log(1-\theta) \in \mathbb{L}_1^{1,2}$
(3) For q-a.e. $(s,x) \in [0,T] \times \mathbb{R}_0$, there is an $\varepsilon_{s,x} \in (0,1)$ such that $\theta_{s,x} < 1 - \varepsilon_{s,x}$.
(4) $Z_T \in L^2(\mathbb{P})$; and $Z_T\{D_{t,0}\log Z_T \mathbf{1}_{\{0\}}(z) + \frac{e^{zD_{t,z}\log Z_T}-1}{z}\mathbf{1}_{\mathbb{R}_0}(z)\} \in L^2(q \times \mathbb{P})$.
(5) $F \in \mathbb{D}^{1,2}$ with $FZ_T \in L^2(\mathbb{P})$; and $Z_T D_{t,z}F + FD_{t,z}Z_T + zD_{t,z}F \cdot D_{t,z}Z_T \in L^2(q \times \mathbb{P})$.
(6) $FH^*_{t,z}, H^*_{t,z}D_{t,z}F \in L^1(\mathbb{P}^*)$, (t,z) -a.e. where $H^*_{t,z} = \exp(zD_{t,z}\log Z_T - \log(1-\theta_{t,z}))$

To show the main theorem, we need the following:

Lemma 4.5.2 *We have*

$$D_{t,0}Z_T = Z_T\left[-\sigma^{-1}u_t - \int_0^T D_{t,0}u_s dW_s^{\mathbb{P}^*} - \int_0^T \int_{\mathbb{R}_0} \frac{D_{t,0}\theta_{s,x}}{1-\theta_{s,x}} \tilde{N}^{\mathbb{P}^*}(ds,dx)\right] \tag{4.5.7}$$

for q-a.e. $(t,z) \in [0,T] \times \{0\}$, $\mathbb{P}$-a.s. and

$$D_{t,z}Z_T = z^{-1}Z_T[\exp(zD_{t,z}\log Z_T) - 1] \quad \text{for } q\text{−a.e. } (t,z) \in [0,T] \times \mathbb{R}_0,\ \mathbb{P}\text{−a.s.,} \tag{4.5.8}$$

where

$$\begin{aligned} D_{t,z}\log Z_T = &- \int_0^T D_{t,z}u_s dW_s^{\mathbb{P}^*} - \frac{1}{2}\int_0^T z(D_{t,z}u_s)^2 ds \\ &+ \int_0^T \int_{\mathbb{R}_0} ((1-\theta_{s,x})D_{t,z}\log(1-\theta_{s,x}) + D_{t,z}\theta_{s,x})\,\nu(dx)ds \\ &+ \int_0^T \int_{\mathbb{R}_0} D_{t,z}\log(1-\theta_{s,x})\tilde{N}^{\mathbb{P}^*}(ds,dx) + z^{-1}\log(1-\theta_{t,z}) \end{aligned} \tag{4.5.9}$$

for q-a.e. $(t,z) \in [0,T] \times \mathbb{R}_0$, $\mathbb{P}$-a.s.

Proof. By conditions (1), (2) and (3) in Assumption 4.5.1, Propositions 4.3.3, 4.3.4 and 4.3.6 imply $\log Z_T \in \mathbb{D}^{1,2}$. Moreover, from (4) in Assumption 4.5.1, Proposition 4.2.9

leads to $Z_T \in \mathbb{D}^{1,2}$,

$$\begin{aligned} D_{t,0}Z_T = Z_T \Bigg[&-D_{t,0}\int_0^T u_s dW_s - \frac{1}{2}D_{t,0}\int_0^T u_s^2 ds \\ &+ D_{t,0}\int_0^T \int_{\mathbb{R}_0} \log(1-\theta_{s,x})\tilde{N}(ds,dx) \\ &+ D_{t,0}\int_0^T \int_{\mathbb{R}_0} (\log(1-\theta_{s,x}) + \theta_{s,x})\nu(dx)ds \Bigg]. \end{aligned} \tag{4.5.10}$$

and

$$D_{t,z}Z_T = \frac{\exp(\log Z_T + zD_{t,z}\log Z_T) - Z_T}{z} = z^{-1}Z_T[\exp(zD_{t,z}\log Z_T) - 1].$$

We next calculate right side of (4.5.10). From assumption (1) in Assumption 4.5.1, Proposition 4.3.4 implies

$$D_{t,0}\int_0^T u_s^2 ds = \int_0^T D_{t,0}u_s^2 ds \tag{4.5.11}$$

and by Proposition 4.3.6,

$$D_{t,0}\int_0^T \int_{\mathbb{R}_0} (\log(1-\theta_{s,x}) + \theta_{s,x})\nu(dx)ds = \int_0^T \int_{\mathbb{R}_0} (D_{t,0}\log(1-\theta_{s,x}) + D_{t,0}\theta_{s,x})\nu(dx)ds. \tag{4.5.12}$$

Since condition (1) in Assumption 4.5.1 holds, by Corollary 4.2.10, we have

$$D_{t,0}u_s^2 = 2u_s D_{t,0}u_s. \tag{4.5.13}$$

We calculate $D_{t,0}\log(1-\theta_{s,x})$. From (3) in Assumption 4.5.1, we have $\theta_{s,x} < 1 - \varepsilon_{s,x}$. We fix $(s,x) \in [0,T] \times \mathbb{R}_0$. We denote

$$l_{s,x}(y) = -\varepsilon_{s,x}^{-1}y + \varepsilon_{s,x}^{-1} - 1 + \log \varepsilon_{s,x}$$

and

$$g_{s,x}(y) = \begin{cases} \log(1-y), & y < 1 - \varepsilon_{s,x} \\ l_{s,x}(y), & y \geq 1 - \varepsilon_{s,x} \end{cases}.$$

Then, $g_{s,x} \in C^1(\mathbb{R})$ and

$$\log(1-\theta_{s,x}) = g_{s,x}(\theta_{s,x}).$$

Moreover, we have $\left|\frac{D_{t,0}\theta_{s,x}}{1-\theta_{s,x}}\right| < \varepsilon_{s,x}^{-1}|D_{t,0}\theta_{s,x}| \in L^2(\lambda \times \mathbb{P})$ by $\frac{1}{1-\theta_{s,x}} < \varepsilon_{s,x}^{-1}$ and $\theta_{s,x} \in \mathbb{D}^{1,2}$. Hence, Proposition 4.2.9 implies that $\log(1-\theta_{s,x}) \in \mathbb{D}_0^{1,2}$ and

$$D_{t,0}\log(1-\theta_{s,x}) = D_{t,0}g_{s,x}(\theta_{s,x}) = g'_{s,x}(\theta_{s,x})D_{t,0}\theta_{s,x} = -\frac{D_{t,0}\theta_{s,x}}{1-\theta_{s,x}}.$$

From condition (1), (2) in Assumption 4.5.1, Proposition 4.3.3 implies

$$D_{t,0}\int_0^T u_s dW_s = \sigma^{-1}u_t + \int_0^T D_{t,0}u_s dW_s \tag{4.5.14}$$

and

$$D_{t,0}\int_0^T\int_{\mathbb{R}_0}\log(1-\theta_{s,x})\tilde{N}(ds,dx) = \int_0^T\int_{\mathbb{R}_0} D_{t,0}\log(1-\theta_{s,x})\tilde{N}(ds,dx). \tag{4.5.15}$$

Hence, by (4.5.10) - (4.5.15), we obtain

$$\begin{aligned}
D_{t,0}Z_T &= Z_T\Bigg[-\sigma^{-1}u_t - \int_0^T D_{t,0}u_s dW_s - \int_0^T u_s D_{t,0}u_s ds \\
&\quad - \int_0^T\int_{\mathbb{R}_0}\frac{D_{t,0}\theta_{s,x}}{1-\theta_{s,x}}\tilde{N}(ds,dx) + \int_0^T\int_{\mathbb{R}_0}\left(-\frac{D_{t,0}\theta_{s,x}}{1-\theta_{s,x}} + D_{t,0}\theta_{s,x}\right)\nu(dx)ds\Bigg] \\
&= Z_T\left[-\sigma^{-1}u_t - \int_0^T D_{t,0}u_s dW_s^{\mathbb{P}^*} - \int_0^T\int_{\mathbb{R}_0}\frac{D_{t,0}\theta_{s,x}}{1-\theta_{s,x}}\tilde{N}^{\mathbb{P}^*}(ds,dx)\right].
\end{aligned}$$

We next calculate $D_{t,z}\log Z_T$.
By conditions (1) and (2) in Assumption 4.5.1, Proposition 4.3.3, Proposition 4.3.4 and Proposition 4.3.6 show that

$$\begin{aligned}
D_{t,z}\log Z_T &= -D_{t,z}\int_0^T u_s dW_s - \frac{1}{2}D_{t,z}\int_0^T u_s^2 ds \\
&\quad + D_{t,z}\int_0^T\int_{\mathbb{R}_0} x^{-1}\log(1-\theta_{s,x})x\tilde{N}(ds,dx) \\
&\quad + D_{t,z}\int_0^T\int_{\mathbb{R}_0}(\log(1-\theta_{s,x})+\theta_{s,x})\nu(dx)ds \\
&= -\int_0^T D_{t,z}u_s dW_s - \frac{1}{2}\int_0^T D_{t,z}(u_s)^2 ds \\
&\quad + \int_0^T\int_{\mathbb{R}_0} D_{t,z}\log(1-\theta_{s,x})\tilde{N}(ds,dx) \\
&\quad + \int_0^T\int_{\mathbb{R}_0}\left(D_{t,z}\log(1-\theta_{s,x}) + D_{t,z}\theta_{s,x}\right)\nu(dx)ds + \frac{\log(1-\theta_{t,z})}{z}.
\end{aligned} \tag{4.5.16}$$

Now we calculate $D_{t,z}(u_s)^2$. Corollary 4.2.10 implies

$$D_{t,z}(u_s)^2 = 2u_s D_{t,z}u_s + z(D_{t,z}u_s)^2, \tag{4.5.17}$$

because, $u \in \mathbb{D}^{1,2}$ and condition (1) in Assumption 4.5.1 hold. From equations (4.5.16) and (4.5.17), we have

$$\begin{aligned}
D_{t,z} \log Z_T = &- \int_0^T D_{t,z} u_s dW_s^{\mathbb{P}^*} - \frac{1}{2} \int_0^T z (D_{t,z} u_s)^2 ds \\
&+ \int_0^T \int_{\mathbb{R}_0} ((1-\theta_{s,x}) D_{t,z} \log(1-\theta_{s,x}) + D_{t,z}\theta_{s,x})\, \nu(dx) ds \\
&+ \int_0^T \int_{\mathbb{R}_0} D_{t,z} \log(1-\theta_{s,x}) \tilde{N}^{\mathbb{P}^*}(ds,dx) + z^{-1} \log(1-\theta_{t,z}).
\end{aligned}$$

□

We next introduce a Clark-Ocone type formula under change of measure for canonical Lévy processes.

Theorem 4.5.3

$$\begin{aligned}
F = \mathbb{E}_{\mathbb{P}^*}[F] &+ \sigma \int_0^T \mathbb{E}_{\mathbb{P}^*}\left[D_{t,0} F - F K_t \middle| \mathcal{F}_{t-} \right] dW_t^{\mathbb{P}^*} \\
&+ \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[F(H_{t,z}^* - 1) + z H_{t,z}^* D_{t,z} F | \mathcal{F}_{t-}] \tilde{N}^{\mathbb{P}^*}(dt,dz), a.s.
\end{aligned}$$

holds, where

$$K_t = \int_0^T D_{t,0} u_s dW_s^{\mathbb{P}^*} + \int_0^T \int_{\mathbb{R}_0} \frac{D_{t,0}\theta_{s,x}}{1-\theta_{s,x}} \tilde{N}^{\mathbb{P}^*}(ds,dx).$$

Proof. First we denote $\Lambda_t := Z_t^{-1} = e^{-\log Z_t}, t \in [0,T]$. Then by the Itô formula (see, e.g., Theorem 9.4 of Di Nunno et al. [20]), we have

$$\begin{aligned}
d\Lambda_t = &\Lambda_{t-} \left(\frac{1}{2} u_t^2 - \int_{\mathbb{R}_0} (\log(1-\theta_{t,z}) + \theta_{t,z}) \nu(dz) \right) dt \\
&+ \Lambda_{t-} u_t dW_t + \frac{1}{2} \Lambda_{t-} u_t^2 dt + \int_{\mathbb{R}_0} \Lambda_{t-} \left(\frac{1}{1-\theta_{t,z}} - 1 \right) \tilde{N}(dt,dz) \\
&+ \int_{\mathbb{R}_0} \left[\Lambda_{t-} \cdot \frac{1}{1-\theta_{t,z}} - \Lambda_{t-} + \Lambda_{t-} \log(1-\theta_{t,z}) \right] \nu(dz) dt \\
= &\Lambda_{t-} \left[u_t^2 dt + u_t dW_t + \int_{\mathbb{R}_0} \frac{\theta_{t,z}^2}{1-\theta_{t,z}} \nu(dz) dt + \int_{\mathbb{R}_0} \frac{\theta_{t,z}}{1-\theta_{t,z}} \tilde{N}(dt,dz) \right] \\
= &\Lambda_{t-} \left[u_t dW_t^{\mathbb{P}^*} + \int_{\mathbb{R}_0} \frac{\theta_{t,z}}{1-\theta_{t,z}} \tilde{N}^{\mathbb{P}^*}(dt,dz) \right].
\end{aligned}$$

Denoting $Y_t := \mathbb{E}_{\mathbb{P}^*}[F|\mathcal{F}_t], t \in [0,T]$, we have $Y_t = \Lambda_t \mathbb{E}[Z_T F | \mathcal{F}_t]$ by condition (5) in Assumption 4.5.1 and the Beyes rule (see, e.g., Lemma 4.7 of Di Nunno et al. [20]). From (5) in Assumption 4.5.1, Corollary 4.2.10 implies that $Z_T F \in \mathbb{D}^{1,2}$. Hence, Lemma 4.2.5

implies that $\mathbb{E}[Z_T F|\mathcal{F}_t] \in \mathbb{D}^{1,2}$ holds. We apply Proposition 4.4.1 to $\mathbb{E}[Z_T F|\mathcal{F}_t]$, then, by Lemma 4.2.5, we have

$$\mathbb{E}[Z_T F|\mathcal{F}_t] = \mathbb{E}[Z_T F] + \int_0^t \int_{\mathbb{R}} \mathbb{E}[D_{s,z}(Z_T F)|\mathcal{F}_{s-}]Q(ds, dz).$$

Denoting $V_t := \mathbb{E}[Z_T F|\mathcal{F}_t]$, we have $Y_t = \Lambda_t V_t$. Itô's product rule implies that

$$\begin{aligned}
dY_t &= \Lambda_{t-} dV_t + V_{t-} d\Lambda_t + d[\Lambda, V]_t \\
&= \Lambda_{t-}[\sigma \mathbb{E}[D_{t,0}(Z_T F)|\mathcal{F}_{t-}]dW_t + \int_{\mathbb{R}_0} \mathbb{E}[D_{t,z}(Z_T F)|\mathcal{F}_{t-}]z\tilde{N}(dt, dz)] \\
&\quad + V_{t-}\Lambda_{t-}\left[u_t dW_t^{\mathbb{P}^*} + \int_{\mathbb{R}_0} \frac{\theta_{t,z}}{1-\theta_{t,z}} \tilde{N}^{\mathbb{P}^*}(dt, dz)\right] \\
&\quad + \Lambda_{t-}[\sigma u_t \mathbb{E}[D_{t,0}(Z_T F)|\mathcal{F}_{t-}] + \int_{\mathbb{R}_0} \frac{\theta_{t,z}}{1-\theta_{t,z}} \mathbb{E}[D_{t,z}(Z_T F)|\mathcal{F}_{t-}]z\nu(dz)]dt \\
&\quad + \Lambda_{t-} \int_{\mathbb{R}_0} \frac{\theta_{t,z}}{1-\theta_{t,z}} \mathbb{E}[D_{t,z}(Z_T F)|\mathcal{F}_{t-}]z\tilde{N}(ds, dz) \\
&= \Lambda_{t-}\mathbb{E}[\sigma D_{t,0}(Z_T F)|\mathcal{F}_{t-}]dW_t^{\mathbb{P}^*} + \Lambda_{t-}\mathbb{E}[Z_T F u_t|\mathcal{F}_{t-}]dW_t^{\mathbb{P}^*} \\
&\quad + \Lambda_{t-} \int_{\mathbb{R}_0} \frac{\mathbb{E}[D_{t,z}(Z_T F)|\mathcal{F}_{t-}]}{1-\theta_{t,z}} z\tilde{N}^{\mathbb{P}^*}(dt, dz) \\
&\quad + \Lambda_{t-} \int_{\mathbb{R}_0} \mathbb{E}\left[Z_T F \frac{\theta_{t,z}}{1-\theta_{t,z}} \middle| \mathcal{F}_{t-}\right] \tilde{N}^{\mathbb{P}^*}(dt, dz).
\end{aligned} \tag{4.5.18}$$

Now we shall calculate $D_{t,0}(Z_T F)$ and $D_{t,z}(Z_T F)$. As for $D_{t,0}(Z_T F)$, by (5) in Assumption 4.5.1, Corollary 4.2.10 yields that

$$D_{t,0}(Z_T F) = F D_{t,0} Z_T + Z_T D_{t,0} F. \tag{4.5.19}$$

Therefore combining (4.5.19) with (4.5.7), we can conclude

$$\begin{aligned}
D_{t,0}(Z_T F) &= F D_{t,0} Z_T + Z_T D_{t,0} F \\
&= F Z_T \left[-\sigma^{-1} u_t - \int_0^T D_{t,0} u_s dW_s^{\mathbb{P}^*} - \int_0^T \int_{\mathbb{R}_0} \frac{D_{t,0}\theta_{s,x}}{1-\theta_{s,x}} \tilde{N}^{\mathbb{P}^*}(ds, dx)\right] + Z_T D_{t,0} F \\
&= Z_T \left[D_{t,0} F - F\left(\sigma^{-1} u_t + K_t\right)\right].
\end{aligned} \tag{4.5.20}$$

Next we calculate $D_{t,z}(Z_T F)$. From condition (5), Corollary 4.2.10 implies that

$$D_{t,z}(Z_T F) = F D_{t,z} Z_T + Z_T D_{t,z} F + z D_{t,z} Z_T \cdot D_{t,z} F. \tag{4.5.21}$$

From (4.5.8),

$$D_{t,z} Z_T = z^{-1} Z_T[(1-\theta_{t,z})H_{t,z}^* - 1]. \tag{4.5.22}$$

Therefore, combining (4.5.21) and (4.5.22), we obtain

$$\begin{aligned}
D_{t,z}(Z_T F) &= z^{-1} Z_T[(1-\theta_{t,z})H_{t,z}^* - 1]F + Z_T D_{t,z} F + Z_T[(1-\theta_{t,z})H_{t,z}^* - 1]D_{t,z} F \\
&= Z_T \left[z^{-1}((1-\theta_{t,z})H_{t,z}^* - 1)F + (1-\theta_{t,z})H_{t,z}^* D_{t,z} F\right].
\end{aligned} \tag{4.5.23}$$

From (4.5.18), (4.5.20), (4.5.23), we arrive at:

$$
\begin{aligned}
dY_t &= \Lambda_{t-}\mathbb{E}\left[Z_T\left[\sigma D_{t,0}F - F\left(u_t + \sigma K_t\right)\right]\Big|\mathcal{F}_{t-}\right] dW_t^{\mathbb{P}^*} \\
&\quad + \Lambda_{t-}\int_{\mathbb{R}_0}\mathbb{E}\left[Z_T\left[F\left(H^*_{t,z} - \frac{1}{1-\theta_{t,z}}\right) + zH^*_{t,z}D_{t,z}F\right]\Big|\mathcal{F}_{t-}\right]\tilde{N}^{\mathbb{P}^*}(dt,dz) \\
&\quad + \Lambda_{t-}\mathbb{E}[Z_TFu_t|\mathcal{F}_{t-}]dW_t^{\mathbb{P}^*} + \Lambda_{t-}\int_{\mathbb{R}_0}\mathbb{E}\left[Z_TF\frac{\theta_{t,z}}{1-\theta_{t,z}}\Big|\mathcal{F}_{t-}\right]\tilde{N}^{\mathbb{P}^*}(dt,dz) \\
&= \sigma\Lambda_{t-}\mathbb{E}\left[Z_T\left[D_{t,0}F - FK_t\right]\Big|\mathcal{F}_{t-}\right] dW_t^{\mathbb{P}^*} \\
&\quad + \Lambda_{t-}\int_{\mathbb{R}_0}\mathbb{E}\left[Z_T\{F\left(H^*_{t,z} - 1\right) + zH^*_{t,z}D_{t,z}F\}\Big|\mathcal{F}_{t-}\right]\tilde{N}^{\mathbb{P}^*}(dt,dz).
\end{aligned}
$$

From (1) and (2) in Assumption 4.5.1, we have $K_t \in L^2(\mathbb{P})$ t-a.e. Hence, by (5) in Assumption 4.5.1,

$$\mathbb{E}_{\mathbb{P}^*}[|FK_t|] = \mathbb{E}[|FK_t|Z_T] \le (\mathbb{E}[|K_t|^2])^{1/2}(\mathbb{E}[|FZ_T|^2])^{1/2} < \infty.$$

Moreover, from (5) in Assumption 4.5.1, we have $D_{t,0}F \in L^2(\mathbb{P})$ t-a.e. and

$$\mathbb{E}_{\mathbb{P}^*}[|D_{t,0}F|] = \mathbb{E}[|D_{t,0}F|Z_T] \le (\mathbb{E}[|D_{t,0}F|^2])^{1/2}(\mathbb{E}[Z_T^2])^{1/2} < \infty.$$

Then, by (6) in Assumption 4.5.1 and $F, D_{t,0}F, FK_t \in L^1(\mathbb{P}^*)$ t-a.e., the Beyes rule implies

$$dY_t = \sigma\mathbb{E}_{\mathbb{P}^*}\left[D_{t,0}F - FK_t\Big|\mathcal{F}_{t-}\right]dW_t^{\mathbb{P}^*} + \int_{\mathbb{R}_0}\mathbb{E}_{\mathbb{P}^*}[F(H^*_{t,z}-1) + zH^*_{t,z}D_{t,z}F|\mathcal{F}_{t-}]\tilde{N}^{\mathbb{P}^*}(dt,dz). \tag{4.5.24}$$

Since $Y_t = \mathbb{E}_{\mathbb{P}^*}[F|\mathcal{F}_T] = F, Y(0) = \mathbb{E}_{\mathbb{P}^*}[F|\mathcal{F}_0] = \mathbb{E}_{\mathbb{P}^*}[F]$, Integrating equation (4.5.24) gives

$$
\begin{aligned}
F - \mathbb{E}_{\mathbb{P}^*}[F] &= \sigma\int_0^T \mathbb{E}_{\mathbb{P}^*}\left[D_{t,0}F - FK_t\Big|\mathcal{F}_{t-}\right]dW_t^{\mathbb{P}^*} \\
&\quad + \int_0^T\int_{\mathbb{R}_0}\mathbb{E}_{\mathbb{P}^*}[F(H^*_{t,z}-1) + zH^*_{t,z}D_{t,z}F|\mathcal{F}_{t-}]\tilde{N}^{\mathbb{P}^*}(dt,dz).
\end{aligned}
$$

The proof is concluded. □

Remark 4.5.4 *1. If $\sigma = 0$, $u = 0$ and $\nu \neq 0$, then, $zD_{t,z}F = D_{(t,z)}F$, we obtain a COCM for pure jump Lévy processes:*

$$F = \mathbb{E}_{\mathbb{P}^*}[F] + \int_0^T\int_{\mathbb{R}_0}\mathbb{E}_{\mathbb{P}^*}[F(H^*(t,z) - 1) + H^*_{t,z}D_{(t,z)}F|\mathcal{F}_{t-}]\tilde{N}^{\mathbb{P}^*}(dt,dz),$$

where

$$H^*_{t,z} := \exp\left(\int_0^T\int_{\mathbb{R}_0}\left[D_{(t,z)}\theta_{s,x}) + \log\left(1 - \frac{D_{(t,z)}\theta_{s,x}}{1-\theta_{s,x}}\right)(1-\theta_{s,x})\right]\nu(dx)ds \right.$$
$$\left. + \int_0^T\int_{\mathbb{R}_0}\log\left(1 - \frac{D_{(t,z)}\theta_{s,x}}{1-\theta_{s,x}}\right)\tilde{N}^{\mathbb{P}^*}(ds,dx)\right)$$

and $D_{(t,z)}F$ is a Malliavin difference operator for pure jump Lévy functionals defined in Definition 12.2 in Di Nunno et al. [20] (see Definition 4.5.5).

2. *If $\sigma \neq 0$, $\theta = 0$, and $\nu = 0$, then, $D_{t,0}F = \sigma^{-1}D_tF$ and we can derive a COCM for Brownian motions:*

$$F = \mathbb{E}_{\mathbb{P}^*}[F] + \int_0^T \mathbb{E}_{\mathbb{P}^*}\left[D_tF - F\int_0^T D_tu_s dW_s^{\mathbb{P}^*}\middle|\mathcal{F}_t\right]dW_t^{\mathbb{P}^*},$$

where D_tF is a classical Malliavin derivative (see Definition 2.1.3). See also Definition 3.1 in Di Nunno et al. [20].

Definition 4.5.5 (Malliavin difference operator for pure jump Lévy functionals) *For $n \in \mathbb{N}$ and for*

$$h_n \in L^2_{T,\lambda\times\nu,n} := \{h_n : ([0,T]\times\mathbb{R}_0)^n \to \mathbb{R} :$$
$$\|h\|^2_{L^2_{T,\lambda\times\nu,n}} := \int_{([0,T]\times\mathbb{R}_0)^n} |h((t_1,z_1),\cdots,(t_n,z_n))|^2 dt_1\nu(dz_1)\cdots dt_n\nu(z_n) < \infty\},$$

we denote

$$I_n(h_n) := \int_{([0,T]\times\mathbb{R}_0)^n} h((t_1,z_1),\cdots,(t_n,z_n))\tilde{N}(t_1,z_1)\cdots\tilde{N}(t_n,z_n).$$

For $F \in \mathbb{D}^{1,2}_N := \{F = \sum_{n=0}^{\infty} I_n(f_n) \in L^2(\mathbb{P}) : \sum_{n=1}^{\infty} nn!\|f_n\|^2_{L^2_{T,\lambda\times\nu,n}} < \infty\}$, the Malliavin difference operator for pure jump Lévy functionals is defined by

$$D_{(t,z)}F = \sum_{n=1}^{\infty} nI_{n-1}(f_n((t,z),\cdot))$$

, $\lambda\times\nu-$a.e. $(t,z) \in [0,T]\times\mathbb{R}_0, \mathbb{P}$*-a.s.*

Remark 4.5.6 *To see different points, we review a result of Okur [38]. Let us denote $\mathbb{P}^W$ the Gaussian white noise probability measure on $(\Omega_W, \mathcal{F}_T^W)$, where the sample space is the Schwartz space $\mathcal{S}'(\mathbb{R})$ and $\mathcal{F}_t = \sigma\{W(s), s \le t\} \vee \mathcal{N}_1, \forall t \in [0,T]$. We denote P^η the pure jump Levy white noise probability measure on $(\Omega_\eta, \mathcal{F}_T^\eta)$, where the sample space is the Schwartz space $\mathcal{S}'(\mathbb{R})$ and $\mathcal{F}_t = \sigma\{\eta(s) = \int_0^s\int_{\mathbb{R}_0} x\tilde{N}(du,dx), s \le t\} \vee \mathcal{N}_2, \forall t \in [0,T]$. Here $\mathcal{N}_1$ and $\mathcal{N}_2$ denote $\mathbb{P}^W$-null and P^η-null sets respectively. Let $\Omega = \mathcal{S}'(\mathbb{R}) \times \mathcal{S}'(\mathbb{R})$, $\mathcal{F}_T^W \otimes \mathcal{F}_T^\eta$. Then, we have a unique measure on the product σ-algebra such that $\mathbb{P} = \mathbb{P}^W \times \mathbb{P}^\eta$ and*

$$\mathbb{P}(A) = \mathbb{P}^W(A^W)\mathbb{P}^\eta(A^\eta),\ A^W \in \mathcal{F}_T^W,\ A^\eta \in \mathcal{F}_T^\eta,\ A = A^W \times A^\eta.$$

The orthogonal basis for $L^2(\mathbb{P})$ is the family of $\mathbb{K}_\alpha$ with $\|\mathbb{K}\|_{L^2(\mathbb{P})} = \alpha! := \alpha^{(1)}!\alpha^{(2)}!$ and $\mathbb{K}_\alpha := H_{\alpha^{(1)}}(\omega') \cdot K_{\alpha^{(2)}}(\omega'')$, where $(\omega', \omega'') \in \Omega$, $\alpha = (\alpha^{(1)}, \alpha^{(2)})$ and $\{\alpha^{(i)}\}_{i=1,2} \in \mathbb{I}$ are multi-indexes defined in section 2 of Okur [38], H_α and K_α are the orthogonal basis for $L^2(\mathbb{P}^W)$ and $L^2(\mathbb{P}^\eta)$ respectively. Moreover, for all $F \in L^2(\mathbb{P})$, there exist unique constants c_α such that

$$F(\omega) = \sum_{\alpha \in \mathbb{I}^2} c_\alpha \mathbb{K}(\omega)$$

and we have

$$\|F\|^2_{L^2(\mathbb{P})} = \sum_{\alpha \in \mathbb{I}^2} c^2_\alpha \alpha!.$$

For $F \in L^2(\mathbb{P})$ with some condition, Hida Malliavin derivatives are defined as

$$D_t F = \sum_{\alpha \in \mathbb{I}^2} \sum_{i \geq 1} c_\alpha \alpha^{(1)}_i \mathbb{K}_{\alpha^{(1)} - \epsilon^i} e_i(t),$$

and

$$D_{t,x} F = \sum_{\alpha \in \mathbb{I}^2} \sum_{i \geq 1} c_\alpha \alpha^{(2)}_{k(i,j)} e_i(t) p_j(x),$$

where $\epsilon^k = (0, \cdots, 0, 1, 0, \cdots, 0)$ with 1 in the k th position, $k(i,j) = j + \frac{(i+j-2)(i+j-1)}{2}$, $\{e_i(t)\}_{i \geq 0} \subset \mathcal{S}(\mathbb{R})$ are Hermite functions on $\mathbb{R}$ and $p_j(x) = \|l_{j-1}\|^{-1}_{L^2(x^2\nu(dx))} x l_{j-1}(x)$, where $\{l_0, l_1, l_2, \cdots\}$ with $l_0 = 1$ is the orthogonalization of $\{1, x, x^2, \cdots\}$ with respect to inner product of $L^2(x^2\nu(dx))$.

In this setting, Okur derived the following equation:

$$\begin{aligned} F = \mathbb{E}_{\mathbb{P}^*}[F] &+ \int_0^T \mathbb{E}_{\mathbb{P}^*}\left[D_t F - F \int_t^T D_t u_s dW_s^{\mathbb{P}^*} \middle| \mathcal{F}_t \right] dW_t^{\mathbb{P}^*} \\ &+ \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[F(H^* - 1) + H^* D_{t,x} F | \mathcal{F}_t] \tilde{N}^{\mathbb{P}^*}(dt, dx), \end{aligned}$$

for any $F \in L^2(\mathcal{F}_T; \mathbb{P})$, where

$$\begin{aligned} H^* = \exp\Bigg(&\int_t^T \int_{\mathbb{R}_0} \left[D_{t,x}\theta_{s,z} + \log\left(1 - \frac{D_{t,x}\theta_{s,z}}{1 - \theta_{s,z}}\right)(1 - \theta_{s,z}) \right] \nu(dz) ds \\ &+ \int_t^T \int_{\mathbb{R}_0} \log\left(1 - \frac{D_{t,x}\theta_{s,z}}{1 - \theta_{s,z}}\right) \tilde{N}^{\mathbb{P}^*}(ds, dz) \Bigg). \end{aligned}$$

Of course, to show this equation, we need more conditions, for more detail, see Okur [38].

Corollary 4.5.7 *Assume in addition to all assumptions of Theorem 4.5.3, that u and θ are deterministic functions, then we have*

$$F = \mathbb{E}_{\mathbb{P}^*}[F] + \sigma \int_0^T \mathbb{E}_{\mathbb{P}^*}[D_{t,0}F | \mathcal{F}_{t-}] dW_t^{\mathbb{P}^*} + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[D_{t,z}F | \mathcal{F}_{t-}] z \tilde{N}^{\mathbb{P}^*}(dt, dz).$$

Proof. If u and θ are deterministic functions, then we have $D_{t,z}u(s) = 0 = D_{t,z}\theta(s,x)$ and $H^*(t,z) = 1$. Therefore, thanks to Theorem 4.5.3, we can get the claimed equation. $\square$

Remark 4.5.8 *If $F \in \mathbb{D}^{1,2}$, $u \equiv 0$ and $\theta \equiv 0$, then, we can see that assumptions of Theorem 4.4.2 and Assumption 4.5.1 hold and we obtain equation (4.4.6).*

4.5.2 Poincare type inequalities and log-Sobolev type inequalities

We next consider Poincare type inequalities and log-Sobolev type inequalities as Corollary of Theorem 4.5.3.

Corollary 4.5.9 *We assume that $\theta_{t,z} \in [-1,1)$ for $(t,z) \in [0,T] \times \mathbb{R}_0$ is a nonrandom function.*

1. *Under all assumptions of Theorem 4.4.2 and Assumption 4.5.1, we have*

$$\mathbb{E}_{\mathbb{P}^*}[(F - \mathbb{E}_{\mathbb{P}^*}[F])^2] \leq \sigma^2 \int_0^T \mathbb{E}_{\mathbb{P}^*}\left[|D_{t,0}F - FK_t|^2\right] dt$$
$$+ \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[|F(H^*_{t,z} - 1) + zH^*_{t,z}D_{t,z}F|^2]\nu^{\mathbb{P}^*}(dz,dt),$$

   where $\nu^{\mathbb{P}^*}(dz,dt) = (1 + \theta_{t,z})\nu(dz)dt$.

2. *Let $F \in \mathbb{D}^{1,2}$ with $F > \eta$ for some $\eta > 0$ and we assume that $\mathcal{F}_{t-} = \mathcal{F}_t$ for all $t \geq 0$. Moreover, we denote $U_t = \mathbb{E}_{\mathbb{P}^*}[F|\mathcal{F}_t]$ and we assume that $U_t > 0$ and $U_t + \mathbb{E}_{\mathbb{P}^*}[F(H^*_{t,z} - 1) + zH^*_{t,z}D_{t,z}F|\mathcal{F}_t] > 0$. Then, under all assumptions of Theorem 4.4.2 and Assumption 4.5.1, we have*

$$\mathbb{E}_{\mathbb{P}^*}[F\log F] - \mathbb{E}_{\mathbb{P}^*}[F]\log \mathbb{E}_{\mathbb{P}^*}[F] \leq \frac{1}{2}\sigma^2 \int_0^T \mathbb{E}_{\mathbb{P}^*}\left[U_t^{-1}|D_{t,0}F - FK_t|^2\right] dt$$
$$+ \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[U_t^{-1}|F(H^*_{t,z} - 1) + zH^*_{t,z}D_{t,z}F|^2]\nu^{\mathbb{P}^*}(dz,dt).$$

Proof.

1. Theorem 4.5.3 implies that

$$\mathbb{E}_{\mathbb{P}^*}[(F - \mathbb{E}_{\mathbb{P}^*}[F])^2] = \mathbb{E}_{\mathbb{P}^*}\left[\left(\sigma \int_0^T \mathbb{E}_{\mathbb{P}^*}\left[D_{t,0}F - FK_t\middle|\mathcal{F}_{t-}\right] dW_t^{\mathbb{P}^*}\right.\right.$$
$$\left.\left.+ \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[F(H^*_{t,z} - 1) + zH^*_{t,z}D_{t,z}F|\mathcal{F}_{t-}]\tilde{N}^{\mathbb{P}^*}(dt,dz)\right)^2\right]$$
$$= \sigma^2 \int_0^T \mathbb{E}_{\mathbb{P}^*}\left[\mathbb{E}_{\mathbb{P}^*}\left[D_{t,0}F - FK_t\middle|\mathcal{F}_{t-}\right]^2\right] dt$$
$$+ \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[\mathbb{E}_{\mathbb{P}^*}[F(H^*_{t,z} - 1) + zH^*_{t,z}D_{t,z}F|\mathcal{F}_{t-}]^2]\nu^{\mathbb{P}^*}(dz,dt)$$

$$
\begin{aligned}
&\leq \sigma^2 \int_0^T \mathbb{E}_{\mathbb{P}^*}\left[|D_{t,0}F - FK_t|^2\right] dt \\
&+ \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[|F(H^*_{t,z}-1) + zH^*_{t,z}D_{t,z}F|^2]\nu^{\mathbb{P}^*}(dz,dt),
\end{aligned}
$$

where we use the Jensen's inequality and Itô isometry.

2. First we denote $\zeta_t = \mathbb{E}_{\mathbb{P}^*}[D_{t,0}F - FK_t|\mathcal{F}_t]$ and $\xi_{t,z} = \mathbb{E}_{\mathbb{P}^*}[F(H^*_{t,z}-1) + zH^*_{t,z}D_{t,z}F|\mathcal{F}_t]$. The Itô formula (see, e.g., Theorem 9.4 of Di Nunno et al. [20]) implies that

$$
\begin{aligned}
&F\log F - \mathbb{E}_{\mathbb{P}^*}[F]\log \mathbb{E}_{\mathbb{P}^*}[F] \\
&= \sigma \int_0^T (\log U_t + 1)\zeta_t dW_t^{\mathbb{P}^*} + \frac{1}{2}\sigma^2 \int_0^T U_t^{-1}\zeta_t^2 dt \\
&+ \int_0^T \int_{\mathbb{R}_0} \{(U_t + \xi_{t,z})(\log(U_t + \xi_{t,z}) - U_t \log U_t \\
&- (\log U_t + 1)\xi_{t,z}\}\nu^{\mathbb{P}^*}(dz,dt) \\
&+ \int_0^T \int_{\mathbb{R}_0} \{(U_t + \xi_{t,z})(\log(U_t + \xi_{t,z}) - U_t \log U_t\}\tilde{N}^{\mathbb{P}^*}(dt,dz).
\end{aligned}
$$

Then, we obtain

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{P}^*}[F\log F] - \mathbb{E}_{\mathbb{P}^*}[F]\log \mathbb{E}_{\mathbb{P}^*}[F] \\
&= \frac{1}{2}\sigma^2 \mathbb{E}_{\mathbb{P}^*}[\int_0^T U_t^{-1}\zeta_t^2 dt] + \mathbb{E}_{\mathbb{P}^*}[\int_0^T \int_{\mathbb{R}_0} \{(U_t + \xi_{t,z})(\log(U_t + \xi_{t,z}) \\
&- U_t \log U_t - (\log U_t + 1)\xi_{t,z}\}\nu^{\mathbb{P}^*}(dz,dt)] \\
&\leq \frac{1}{2}\sigma^2 \mathbb{E}_{\mathbb{P}^*}[\int_0^T U_t^{-1}\zeta_t^2 dt] + \mathbb{E}_{\mathbb{P}^*}[\int_0^T \int_{\mathbb{R}_0} U_t^{-1}\xi_{t,z}^2 \nu^{\mathbb{P}^*}(dz,dt)] \\
&\leq \frac{1}{2}\sigma^2 \int_0^T \mathbb{E}_{\mathbb{P}^*}\left[U_t^{-1}|D_{t,0}F - FK_t|^2\right] dt \\
&+ \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[U_t^{-1}|F(H^*_{t,z}-1) + zH^*_{t,z}D_{t,z}F|^2]\nu^{\mathbb{P}^*}(dz,dt),
\end{aligned}
$$

where we use the Jensen's inequality and the following inequality:

$$
(x+y)\log(x+y) - x\log x - y(1+\log x) \leq \frac{y^2}{x^2}, x > 0, x+y > 0.
$$

□

Remark 4.5.10 *1. Assume in addition to all assumptions of Corollary 4.5.9, that u and θ are deterministic functions, then, we obtain a Poincare's inequality for Lévy functionals on $\mathbb{P}^*$:*

$$\mathbb{E}_{\mathbb{P}^*}[(F - \mathbb{E}_{\mathbb{P}^*}[F])^2] \leq \sigma^2 \int_0^T \mathbb{E}_{\mathbb{P}^*}\left[|D_{t,0}F|^2\right] dt + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[|zD_{t,z}F|^2]\nu^{\mathbb{P}^*}(dz, dt).$$

2. *If $F \in \mathbb{D}^{1,2}$, $u \equiv 0$ and $\theta \equiv 0$, then, we can see that all assumptions of Theorem 4.4.2 and Assumption 4.5.1 hold and we obtain a Poincare's inequality for Lévy functionals:* $\mathbb{E}[(F - \mathbb{E}[F])^2] \leq \int_0^T \int_{\mathbb{R}} \mathbb{E}[|D_{t,z}F|^2]q(dt, dz)$.
3. *Assume in addition to all assumptions of Corollary 4.5.9, that u and θ are deterministic functions, then, we obtain a logarithmic Sobolev inequality for Lévy functionals on $\mathbb{P}^*$*

$$\mathbb{E}_{\mathbb{P}^*}[F \log F] - \mathbb{E}_{\mathbb{P}^*}[F] \log \mathbb{E}_{\mathbb{P}^*}[F] \leq \frac{1}{2}\sigma^2 \int_0^T \mathbb{E}_{\mathbb{P}^*}\left[U_t^{-1}|D_{t,0}F|^2\right] dt + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[U_t^{-1}|zD_{t,z}F|^2]\nu^{\mathbb{P}^*}(dz, dt).$$

4. *Assume in addition to all assumptions of Corollary 4.5.9, that $u \equiv 0$ and $\theta \equiv 0$, then, we obtain a logarithmic Sobolev inequality for Lévy functionals:*

$$\mathbb{E}[F \log F] - \mathbb{E}[F] \log \mathbb{E}[F] \leq \frac{1}{2}\sigma^2 \int_0^T \mathbb{E}\left[U_t^{-1}|D_{t,0}F|^2\right] dt + \int_0^T \int_{\mathbb{R}_0} [U_t^{-1}|zD_{t,z}F|^2]\nu(dz, dt).$$

Chapter 5

Local risk minimization for Lévy markets

In this chapter, we obtain explicit representations of locally risk-minimizing by using the results of previous chapter. For incomplete market models whose asset price is described by a solution to a stochastic differential equation driven by a Lévy process, we derive general formulas of locally risk-minimizing including Malliavin derivatives; and calculate its concrete expressions for call options, Asian options and lookback options.

The content of this chapter is based on Arai and Suzuki [5].

5.1 Introduction

In this chapter, we obtain explicit representations of locally risk-minimizing by using Malliavin calculus for Lévy processes given by previous chapter.

Locally risk-minimizing (LRM, for short) is a well-known hedging method for contingent claims in a quadratic way. Theoretical aspects of LRM have been developed to a high degree. On the other hand, the necessity of researches on its explicit representations has been increasing. From this insight, we aim to obtain explicit representations of LRM for incomplete market models whose asset price process is described by a solution to a stochastic differential equation (SDE, for short) driven by a Lévy process, as a typical framework of incomplete market models. In particular, we use Malliavin calculus for Lévy processes to achieve our purpose.

LRM has more than two decades history. There is so much literature on this topic. Among other things, Schweizer [45] and [46] are useful to understand an outline. LRM has an intimate relationship with Föllmer-Schweizer decomposition (FS decomposition, for short), which is a kind of orthogonal decomposition of a random variable into a stochastic integration and an orthogonal martingale. As the first step, we focus on deriving a representation of FS decomposition under some mild conditions by using the martingale representation theorem. In order to compute its explicit expressions, we use Malliavin calculus. Note that we adopt the approach, undertaken by Solé, Utzet and Vives [49], of Malliavin calculus for Lévy processes on canonical Lévy space. As a result, using the Clark-Ocone type formula under change of measure shown by Suzuki [50], [51] (see previous chapter), we will formulate general representations of LRM including Malliavin derivatives of the claim to be hedged.

In the second half of this chapter, we derive formulas on representations of LRM for three typical options. Firstly, we shall study call options, whose payoff is not smooth as a function of the asset price at the maturity. Thus, the chain rule is not available to calculate Malliavin derivatives for call options. Instead, we use the mollifier approximation. Moreover, we illustrate a concrete expression of LRM for the models whose asset price process is a solution to an SDE with deterministic coefficients. Next, Asian options will be discussed. Thirdly, we shall deal with lookback options, whose payoff is depending on the running maximum of the asset price process. Actually, we need complicated calculations to get Malliavin derivatives of the running maximum. For lookback options, we shall focus only on the exponential Lévy case; and derive Malliavin derivatives by using an approximation method.

Summarizing the above, our main contribution is threefold as follows:

1. formulating representations of LRM with Malliavin derivatives for Lévy markets,
2. illustrating how to calculate Malliavin derivatives for non-smooth functions of a random variable, and the running maximum of processes by using approximation methods.
3. introducing concrete representations of LRM of call options, Asian options and lookback options for Lévy markets.

This chapter is structured as follows: In Section 5.2, we prepare some terminologies; and give model descriptions, mathematical preliminaries and standing assumptions. We also introduce in Section 5.2 examples satisfying our standing assumptions. General representations of LRM are introduced in Section 5.3. Call options, Asian options and lookback options are studied in Sections 5.4, 5.5 and 5.6, respectively. Section 5.7 is devoted to concluding remarks.

5.2 Preliminaries

5.2.1 Model description

We consider, throughout this chapter, a financial market being composed of one risk-free asset and one risky asset with finite time horizon T. For simplicity, we assume that the interest rate of the market is given by 0, that is, the price of the risk-free asset is 1 at all times. The fluctuation of the risky asset is assumed to be given by a solution to the following stochastic differential equation (SDE, for short) on canonical space $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{t\in[0,T]})$:

$$dS_t = S_{t-}\left[\alpha_t dt + \beta_t dW_t + \int_{\mathbb{R}_0} \gamma_{t,z}\widetilde{N}(dt, dz)\right], \quad S_0 > 0, \tag{5.2.1}$$

where α, β and γ are predictable processes. Recall that γ is a stochastic process measurable with respect to the σ-algebra generated by $A \times (s, u] \times B$, $A \in \mathcal{F}_s$, $0 \leq s < u \leq T$, $B \in \mathcal{B}(\mathbb{R}_0)$. Now, we assume the following:

Assumption 5.2.1 *1. (5.2.1) has a solution S satisfying the so-called structure condition (SC, for short). That is, S is a special semimartingale with the canonical decomposition $S = S_0 + M + A$ such that*

$$\left\| [M]_T^{1/2} + \int_0^T |dA_s| \right\|_{L^2(\mathbb{P})} < \infty, \tag{5.2.2}$$

where $dM_t = S_{t-}(\beta_t dW_t + \int_{\mathbb{R}_0} \gamma_{t,z} \widetilde{N}(dt, dz))$ and $dA_t = S_{t-}\alpha_t dt$. Moreover, defining a process $\lambda_t := \dfrac{\alpha_t}{S_{t-}(\beta_t^2 + \int_{\mathbb{R}_0} \gamma_{t,z}^2 \nu(dz))}$, we have $A = \int \lambda d\langle M \rangle$. Thirdly, the mean-variance trade-off process $K_t := \int_0^t \lambda_s^2 d\langle M \rangle_s$ is finite, that is, K_T is finite $\mathbb{P}$-a.s.

2. *$\gamma_{t,z} > -1$, (t, z, ω)-a.e., that is, $\mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} \mathbf{1}_{\{\gamma_{t,z} \le -1\}} \nu(dz) dt\right] = 0$.*

Remark 5.2.2 *1. The SC is closely related to the no-arbitrage condition. For more details on the SC, see Schweizer [45] and [46].*

2. *The process K as well as A is continuous.*
3. *(5.2.2) implies that $\sup_{t \in [0,T]} |S_t| \in L^2(\mathbb{P})$ by Theorem V.2 of Protter [40].*
4. *Condition 2 ensures that $S_t > 0$ for any $t \in [0, T]$.*

5.2.2 Locally risk-minimizing

We define locally risk-minimizing (LRM, for short) for a contingent claim $F \in L^2(\mathbb{P})$. The following definition is based on Theorem 1.6 of Schweizer [46].

Definition 5.2.3 *1. Θ_S denotes the space of all $\mathbb{R}$-valued predictable processes ξ satisfying*

$$\mathbb{E}\left[\int_0^T \xi_t^2 d\langle M \rangle_t + (\int_0^T |\xi_t dA_t|)^2\right] < \infty$$

.

2. *An L^2-strategy is given by a pair $\varphi = (\xi, \eta)$, where $\xi \in \Theta_S$ and η is an adapted process such that $V(\varphi) := \xi S + \eta$ is a right continuous process with $\mathbb{E}[V_t^2(\varphi)] < \infty$ for every $t \in [0, T]$. Note that ξ_t (resp. η_t) represents the amount of units of the risky asset (resp. the risk-free asset) an investor holds at time t.*
3. *For $F \in L^2(\mathbb{P})$, the process $C^F(\varphi)$ defined by $C_t^F(\varphi) := F1_{\{t=T\}} + V_t(\varphi) - \int_0^t \xi_s dS_s$ is called the cost process of $\varphi = (\xi, \eta)$ for F.*
4. *An L^2-strategy φ is said locally risk-minimizing for F if $V_T(\varphi) = 0$ and $C^F(\varphi)$ is a martingale orthogonal to M, that is, $[C^F(\varphi), M]$ is a uniformly integrable martingale.*

The above definition of LRM is a simplified version, since the original one, introduced in Schweizer [45] and [46], is rather complicated

Now, we focus on a representation of LRM. To this end, we define Föllmer-Schweizer decomposition (FS decomposition, for short).

Definition 5.2.4 *An $F \in L^2(\mathbb{P})$ admits a Föllmer-Schweizer decomposition if it can be described by*

$$F = F_0 + \int_0^T \xi_t^F dS_t + L_T^F, \tag{5.2.3}$$

where $F_0 \in \mathbb{R}$, $\xi^F \in \Theta_S$ and L^F is a square-integrable martingale orthogonal to M with $L_0^F = 0$.

Proposition 5.2 of Schweizer [46] shows the following:

Proposition 5.2.5 (Proposition 5.2 of Schweizer [46]) *Under Assumption 5.2.1, an LRM $\varphi = (\xi, \eta)$ for F exists if and only if F admits an FS decomposition; and its relationship is given by*

$$\xi_t = \xi_t^F, \quad \eta_t = F_0 + \int_0^t \xi_s^F dS_s + L_t^F - F1_{\{t=T\}} - \xi_t^F S_t.$$

As a result, it suffices to obtain a representation of ξ^F in (5.2.3) in order to obtain LRM. Henceforth, we identify ξ^F with LRM. To this end, we consider the process $Z := \mathcal{E}(-\int \lambda dM)$, where $\mathcal{E}(Y)$ represents the stochastic exponential of Y, that is, Z is a solution to the SDE $dZ_t = -\lambda_t Z_{t-} dM_t$. In addition to Assumption 5.2.1, we suppose the following:

Assumption 5.2.6 *Z is a positive square integrable martingale; and $Z_T F \in L^2(\mathbb{P})$.*

A martingale measure $\mathbb{P}^* \sim \mathbb{P}$ is called minimal if any square-integrable $\mathbb{P}$-martingale orthogonal to M remains a martingale under $\mathbb{P}^*$. We can see the following:

Lemma 5.2.7 *Under Assumption 5.2.1, if Z is a positive square integrable martingale, then a minimal martingale measure $\mathbb{P}^*$ exists with $d\mathbb{P}^* = Z_T d\mathbb{P}$.*

Proof. Since $d(ZS) = S_- dZ + Z_- dM + Z_- \lambda d\langle M \rangle - Z_- \lambda d[M]$, the product process ZS is a $\mathbb{P}$-local martingale. So that, defining a probability measure $\mathbb{P}^*$ as $d\mathbb{P}^* = Z_T d\mathbb{P}$, we have that S is a $\mathbb{P}^*$-martingale, since $\sup_{t \in [0,T]} |S_t|$ and Z_T are in $L^2(\mathbb{P})$. Next, for any L a square-integrable $\mathbb{P}$-martingale with null at 0 orthogonal to M, LZ is a $\mathbb{P}$-local martingale. By the square integrability of L, L remains a martingale under $\mathbb{P}^*$. Thus, $\mathbb{P}^*$ is a minimal martingale measure. □

Example 5.2.8 *We introduce a model framework under which Assumption 5.2.1 is satisfied, and Z is a positive square integrable martingale. We consider the following three conditions:*

1. *$\gamma_{t,z} > -1$, (t, z, ω)-a.e.*
2. *$\sup_{t \in [0,T]} (|\alpha_t| + \beta_t^2 + \int_{\mathbb{R}_0} \gamma_{t,z}^2 \nu(dz)) < C$ for some $C > 0$.*
3. *There exists an $\varepsilon > 0$ such that*

$$\frac{\alpha_t \gamma_{t,z}}{\beta_t^2 + \int_{\mathbb{R}_0} \gamma_{t,z}^2 \nu(dz)} < 1 - \varepsilon \quad \text{and} \quad \beta_t^2 + \int_{\mathbb{R}_0} \gamma_{t,z}^2 \nu(dz) > \varepsilon, (t, z, \omega)\text{-a.e.}$$

The above condition 2 ensures the existence of a unique solution S to (5.2.1) satisfying

$$\sup_{t\in[0,T]} |S_t| \in L^2(\mathbb{P})$$

by Theorem 117 of Situ [48]. The first condition of Assumption 5.2.1 is seen as follows: Firstly, we have $\left\| \int_0^T |dA_s| \right\|^2_{L^2(\mathbb{P})} \leq C^2T^2\mathbb{E}[\sup_{t\in[0,T]} |S_t|^2] < \infty$*. Next, by the Burkholder-Davis-Gundy inequality, there exists a* $C > 0$ *such that*

$$\begin{aligned}\mathbb{E}[[M]_T] &\leq C\mathbb{E}\left[\sup_{t\in[0,T]} |M_t|^2\right] \\ &\leq C\left\{\mathbb{E}\left[\sup_{t\in[0,T]} |S_t|^2\right] + |S_0|^2 + \mathbb{E}\left[\sup_{t\in[0,T]} |A_t|^2\right]\right\} < \infty\end{aligned}$$

Thus, all conditions of Assumption 5.2.1 are satisfied.

On the other hand, the above condition 3 guarantees the positivity of Z. Noting that Z is a solution to $dZ_t = -\lambda_t Z_{t-} dM_t$*, we have* $\sup_{t\in[0,T]} |Z_t| \in L^2(\mathbb{P})$ *by using Theorem 117 of Situ [48] again. In addition, since* $\mathbb{E}[\int_0^T \lambda_t^2 d[M]_t] < \infty$ *by conditions 2 and 3, the process* $-\int_0^\cdot \lambda_s dM_s$ *is a square integrable martingale by Lemma on p.171 of Protter [40]. Thus, the process* $-\int_0^\cdot \lambda_s Z_{s-} dM_s$ *is a local martingale, that is, so is Z. Theorem I.51 of Protter [40] implies that Z is a square integrable martingale. Hence, a minimal martingale measure exists by Lemma 5.2.7.*

5.2.3 Barndorff-Nielsen and Shephard model

We introduce what we call Barndorff-Nielsen and Shephard model as one more example which satisfies Assumption 5.2.1 and the square integrable martingale property of Z. This is an Ornstein-Uhlenbeck type stochastic volatility model, undertaken by Barndorff-Nielsen and Shephard [7], [8]. Let H be a subordinator without drift, that is, a non-decreasing, pure jump and no diffusion component Lévy process with $H_0 = 0$. Note that its Lévy measure ν satisfies $\nu((-\infty, 0)) = 0$ and $\int_0^\infty (z \wedge 1)\nu(dz) < \infty$ by Proposition 3.10 of Cont and Tankov [15]. In addition, we assume that $\int_0^\infty z^2\nu(dz) < \infty$, that is, the square integrability of H. Suppose that the underlying Lévy process X is given as $X = W + \widetilde{H}$, where $\widetilde{H}$ is the compensated process of H. Now, we define a process Σ^2 as a solution to the following SDE:

$$\Sigma_t^2 = \Sigma_0^2 - R\int_0^t \Sigma_s^2 ds + H_t,$$

where $\Sigma_0^2 > 0$ and $R > 0$. By simple calculations, we have $\Sigma_t^2 = e^{-Rt}\Sigma_0^2 + \int_0^t e^{-R(t-s)}dH_s$. In addition, we define

$$L_t := \mu t - \frac{1}{2}\int_0^t \Sigma_s^2 ds + \int_0^t \Sigma_s dW_s + \rho H_t,$$

where $\mu \in \mathbb{R}$ and $\rho \leq 0$. Note that we restrict the coefficient of the second term to $-\frac{1}{2}$ for the sake of simplicity. Now, the asset price process S is assumed to be given by $S_t = S_0 \exp(L_t)$ with $S_0 > 0$, that is, a solution to the following SDE:

$$dS_t = S_{t-}\left\{\alpha dt + \Sigma_t dW_t + \int_{\mathbb{R}_0}(e^{\rho z}-1)\widetilde{N}(dt,dz)\right\}, \tag{5.2.4}$$

where $\alpha := \mu + \int_{\mathbb{R}_0}(e^{\rho z}-1)\nu(dz)$. Note that the SDE (5.2.4) does not satisfy condition 2 of Example 5.2.8. The goal of this subsection is to confirm that the above model satisfies Assumption 5.2.1 and that Z is a positive square integrable martingale under the following additional assumptions:

Assumption 5.2.9 *1.* $\int_1^\infty \exp\left\{2\frac{1-e^{-RT}}{R}z\right\}\nu(dz) < \infty$.

2. $\alpha > 0$ *or* $e^{-RT}\sigma_0^2 + \int_{\mathbb{R}_0}(e^{\rho z}-1)^2\nu(dz) > |\alpha|$.

Remark 5.2.10 *There are two typical examples of the Barndorff-Nielsen and Shephard models. One is the case where Σ_t^2 follows an inverse Gaussian distribution, that is, the process Σ^2 is given as an IG-OU process. The corresponding Lévy measure is given as*

$$\nu(dz) = \frac{a}{2\sqrt{2\pi}}z^{-\frac{3}{2}}(1+b^2z)\exp\left\{-\frac{1}{2}b^2z\right\}\mathbf{1}_{\{z>0\}}dz,$$

where a and b are positive constants. Whenever $\frac{1}{2}b^2 > 2\frac{1-e^{-RT}}{R}$, Condition 1 of Assumption 5.2.9 is satisfied as well as $\int_0^\infty z^2\nu(dz) < \infty$.

The other is the Gamma-OU case. In this case, Σ_t^2 follows a Gamma distribution; and $\nu(dz)$ is given as $\nu(dz) = abe^{-bz}\mathbf{1}_{\{z>0\}}dz$ for $a > 0$ and $b > 0$. If $b > 2\frac{1-e^{-RT}}{R}$, then condition 1 of Assumption 5.2.9 is satisfied. For more details, see Schoutens [42].

As for Assumption 5.2.1, it suffices to see $\mathbb{E}\left[\sup_{t\in[0,T]}|S_t|^2\right] < \infty$ by the same manner as Example 5.2.8. On the other hand, the second condition of Assumption 5.2.9 ensures the positivity of Z. Since $\mathbb{E}[\int_0^T \lambda_t^2 d[M]_t] < \infty$, the square integrable martingale property of Z is shown by the same way as Example 5.2.8.

Lemma 5.2.11 $\mathbb{E}\left[\sup_{t\in[0,T]}|S_t|^2\right] < \infty$.

Proof. *Step 1.* Denoting, for $t \in [0,T]$

$$\begin{aligned}\widehat{M}_t &:= \int_0^t \Sigma_s dW_s - \frac{1}{2}\int_0^t \Sigma_s^2 ds + \rho H_t + t\int_{\mathbb{R}_0}[-e^{\rho z}+1]\nu(dz)\\ &= \int_0^t \Sigma_s dW_s - \frac{1}{2}\int_0^t \Sigma_s^2 ds + \rho\int_0^t\int_{\mathbb{R}_0} z\widetilde{N}(ds,dz) + t\int_{\mathbb{R}_0}[\rho z - e^{\rho z}+1]\nu(dz),\end{aligned}$$

we see that $e^{\widehat{M}}$ is a martingale. From the view of Theorem 1.4 of Ishikawa [25], we have only to make sure the following three conditions:

(1) $\int_0^\infty (1-e^{\rho z})^2 \nu(dz) < \infty$,
(2) $\int_0^\infty (\rho z e^{\rho z} + 1 - e^{\rho z})\nu(dz) < \infty$,
(3) $\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \Sigma_s^2 ds\right)\right] < \infty$.
Since $1-e^{\rho z} \leq |\rho| z$ for any $z > 0$, we have $\int_0^\infty (1-e^{\rho z})^2 \nu(dz) \leq \int_0^1 \rho^2 z^2 \nu(dz) + \int_1^\infty \nu(dz) < \infty$; and $\int_0^\infty (\rho z e^{\rho z} + 1 - e^{\rho z})\nu(dz) \leq \int_0^\infty (1-e^{\rho z})\nu(dz) \leq \int_0^\infty |\rho| z \nu(dz) < \infty$. As for (3), setting $\mathcal{B}(t) := \frac{1}{R}(1-e^{-Rt})$ for $t \in [0,T]$, we have

$$\begin{aligned}
\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \Sigma_s^2 ds\right)\right] &= \mathbb{E}\left[\exp\left(\frac{1}{2}\Sigma_0^2 \mathcal{B}(T) + \frac{1}{2}\int_0^T \mathcal{B}(T-s) dH_s\right)\right] \\
&\leq \exp\left(\frac{1}{2}\Sigma_0^2 \mathcal{B}(T)\right)\mathbb{E}\left[\exp\left(\frac{\mathcal{B}(T)H_T}{2}\right)\right].
\end{aligned}$$

By Proposition 3.14 of Cont and Tankov [15], Assumption 5.2.9 ensures $\mathbb{E}\left[\exp\left(\frac{\mathcal{B}(T)H_T}{2}\right)\right] < \infty$.

Step 2. Next, we see $\mathbb{E}[e^{2\widehat{M}_T}] < \infty$. We have

$$\begin{aligned}
2\widehat{M}_T &= 2\int_0^T \Sigma_s dW_s - \int_0^T \Sigma_s^2 ds + 2\rho \int_0^T \int_{\mathbb{R}_0} z \widetilde{N}(ds,dz) + 2T \int_{\mathbb{R}_0} [\rho z - e^{\rho z} + 1]\nu(dz) \\
&= Y_T + \mathcal{B}(T)\Sigma_0^2 + \int_0^T \int_{\mathbb{R}_0} [e^{g(s)z} - 2e^{\rho z} + 1]\nu(dz)ds,
\end{aligned}$$

where $g(s) := \mathcal{B}(T-s) + 2\rho$ and

$$Y_t := 2\int_0^t \Sigma_s dW_s - 2\int_0^t \Sigma_s^2 ds + \int_0^t \int_{\mathbb{R}_0} g(s) z \widetilde{N}(ds,dz) + \int_0^t \int_{\mathbb{R}_0} [g(s)z - e^{g(s)z} + 1]\nu(dz)ds.$$

Because $2\rho \leq g(s) \leq \mathcal{B}(T) + 2\rho$ for any $s \in [0,T]$,

$$|1-e^{g(s)z}| \leq \begin{cases} z(e^{g(s)}-1), & \text{if } g(s) \geq 0, z \in (0,1), \\ e^{g(s)z}, & \text{if } g(s) \geq 0, z \geq 1, \\ -g(s)z, & \text{if } g(s) < 0, z > 0, \end{cases}$$

and Assumption 5.2.9, we have $\int_0^T \int_{\mathbb{R}_0} |e^{g(s)z} - 1|\nu(dz)ds < \infty$. Moreover, we have $\int_0^\infty (1-e^{\rho z})\nu(dz) < \infty$. We have then $\mathbb{E}[e^{2\widehat{M}_T}] < \infty$ if $\mathbb{E}[e^{Y_T}] = 1$.

Step 3. We show $\mathbb{E}[e^{Y_T}] = 1$. By Theorem 1.4 of Ishikawa [25], it suffices to see the following:
(4) $\int_0^T \int_0^\infty \left\{(1-e^{g(s)z})^2 + g(s)^2 z^2 + |g(s) z e^{g(s)z} + 1 - e^{g(s)z}|\right\} \nu(dz)ds < \infty$,
(5) $\mathbb{E}\left[\exp\left(2\int_0^T \Sigma_s^2 ds\right)\right] < \infty$.
(4) is reduced by the same sort argument as Step 2 and

$$|g(s) z e^{g(s)z}| \leq \begin{cases} g(s) z e^{g(s)}, & \text{if } g(s) \geq 0, z \in (0,1), \\ e^{2g(s)z}, & \text{if } g(s) \geq 0, z \geq 1, \\ -g(s)z, & \text{if } g(s) < 0, z > 0. \end{cases}$$

As for (5), Assumption 5.2.9 and the same argument as Step 1 yield

$$\mathbb{E}\left[\exp\left(2\int_0^T \Sigma_s^2 ds\right)\right] \leq \exp(2\Sigma_0^2 \mathcal{B}(T))\mathbb{E}\left[\exp(2\mathcal{B}(T)H_T)\right] < \infty.$$

Step 4. Since we have $2L_t = 2\mu t + 2\widehat{M}_t + 2t\int_{\mathbb{R}_0}(e^{\rho z}-1)\nu(dz) \leq 2(\mu \vee 0)T + 2\widehat{M}_t$, the Doob inequality yields

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}|S_t|^2\right] &= S_0^2\mathbb{E}\left[\sup_{t\in[0,T]} e^{2L_t}\right] \leq S_0^2 e^{2(\mu\vee 0)T}\mathbb{E}\left[\sup_{t\in[0,T]} e^{2\widehat{M}_t}\right] \\ &\leq 4S_0^2 e^{2(\mu\vee 0)T}\mathbb{E}\left[e^{2\widehat{M}_T}\right] < \infty\end{aligned}$$

by Steps 1-3. □

5.3 Representation results for LRM

In this section, we focus on representations of LRM ξ^F for claim F. First of all, we study it through the martingale representation theorem.

5.3.1 Approach based on the martingale representation theorem

Throughout this subsection, we assume Assumptions 5.2.1 and 5.2.6. Let $\mathbb{P}^*$ be a minimal martingale measure, that is, $d\mathbb{P}^* = Z_T d\mathbb{P}$ holds. The martingale representation theorem (see, e.g. Proposition 9.4 of Cont and Tankov [15]) provides

$$Z_T F = \mathbb{E}_{\mathbb{P}^*}[F] + \int_0^T g_t^0 dW_t + \int_0^T \int_{\mathbb{R}_0} g_{t,z}^1 \widetilde{N}(dt,dz)$$

for some predictable processes g_t^0 and $g_{t,z}^1$. By the same sort of calculations as the proof of Theorem 4.5.3, we have

$$\begin{aligned}F &= \mathbb{E}_{\mathbb{P}^*}[F] + \int_0^T \frac{g_t^0 + \mathbb{E}[Z_T F|\mathcal{F}_{t-}]u_t}{Z_{t-}} dW_t^{\mathbb{P}^*} \\ &\quad + \int_0^T \int_{\mathbb{R}_0} \frac{g_{t,z}^1 + \mathbb{E}[Z_T F|\mathcal{F}_{t-}]\theta_{t,z}}{Z_{t-}(1-\theta_{t,z})} \widetilde{N}^{\mathbb{P}^*}(dt,dz) \\ &=: \mathbb{E}_{\mathbb{P}^*}[F] + \int_0^T h_t^0 dW_t^{\mathbb{P}^*} + \int_0^T \int_{\mathbb{R}_0} h_{t,z}^1 \widetilde{N}^{\mathbb{P}^*}(dt,dz)\end{aligned}$$

where $u_t := \lambda_t S_{t-}\beta_t$, $\theta_{t,z} := \lambda_t S_{t-}\gamma_{t,z}$, $dW_t^{\mathbb{P}^*} := dW_t + u_t dt$ and $\widetilde{N}^{\mathbb{P}^*}(dt,dz) := \widetilde{N}(dt,dz) + \theta_{t,z}\nu(dz)dt$. Girsanov's theorem implies that $W^{\mathbb{P}^*}$ and $\widetilde{N}^{\mathbb{P}^*}$ are a Brownian motion and the compensated Poisson random measure of N under $\mathbb{P}^*$, respectively. Additionally, we assume that

$$\mathbb{E}\left[\int_0^T \left\{(h_t^0)^2 + \int_{\mathbb{R}_0}(h_{t,z}^1)^2\nu(dz)\right\}dt\right] < \infty. \tag{5.3.1}$$

Denoting $i^0_t := h^0_t - \xi_t S_{t-}\beta_t$, $i^1_{t,z} := h^1_{t,z} - \xi_t S_{t-}\gamma_{t,z}$, and

$$\xi_t := \frac{\lambda_t}{\alpha_t}\{h^0_t\beta_t + \int_{\mathbb{R}_0} h^1_{t,z}\gamma_{t,z}\nu(dz)\}, \tag{5.3.2}$$

we can see

$$i^0_t\beta_t + \int_{\mathbb{R}_0} i^1_{t,z}\gamma_{t,z}\nu(dz) = 0$$

for any $t \in [0,T]$, which implies $i^0_t u_t + \int_{\mathbb{R}_0} i^1_{t,z}\theta_{t,z}\nu(dz) = 0$. We have then

$$\begin{aligned} F - \mathbb{E}_{\mathbb{P}^*}[F] - \int_0^T \xi_t dS_t &= \int_0^T i^0_t dW^{\mathbb{P}^*}_t + \int_0^T \int_{\mathbb{R}_0} i^1_{t,z}\widetilde{N}^{\mathbb{P}^*}(dt,dz) \\ &= \int_0^T i^0_t dW_t + \int_0^T \int_{\mathbb{R}_0} i^1_{t,z}\widetilde{N}(dt,dz). \end{aligned}$$

The following lemma implies that $L^F_t := \mathbb{E}[F - \mathbb{E}_{\mathbb{P}^*}[F] - \int_0^T \xi_s dS_s|\mathcal{F}_t]$ is a square integrable martingale orthogonal to M with $L^F_0 = 0$.

Lemma 5.3.1 *Under Assumptions 5.2.1 and 5.2.6, and (5.3.1), we have*

$$\mathbb{E}\left[\int_0^T (i^0_t)^2 dt + \int_0^T \int_{\mathbb{R}_0} (i^1_{t,z})^2\nu(dz)dt\right] < \infty.$$

Proof. Noting that $\frac{\beta_t^2}{\beta_t^2 + \int_{\mathbb{R}_0}\gamma^2_{t,x}\nu(dx)}$ and $\frac{\int_{\mathbb{R}_0}\gamma^2_{t,x}\nu(dx)}{\beta_t^2 + \int_{\mathbb{R}_0}\gamma^2_{t,x}\nu(dx)}$ are less than 1, we have

$$\begin{aligned} \mathbb{E}\left[\int_0^T \xi_t^2 S_{t-}^2\beta_t^2 dt\right] &\leq 2\mathbb{E}\left[\int_0^T \frac{\beta_t^4(h^0_t)^2 + \beta_t^2\left(\int_{\mathbb{R}_0} h^1_{t,x}\gamma_{t,x}\nu(dx)\right)^2}{\left(\beta_t^2 + \int_{\mathbb{R}_0}\gamma^2_{t,x}\nu(dx)\right)^2}dt\right] \\ &\leq 2\mathbb{E}\left[\int_0^T \frac{\beta_t^4(h^0_t)^2 + \beta_t^2\int_{\mathbb{R}_0}(h^1_{t,x})^2\nu(dx)\int_{\mathbb{R}_0}\gamma^2_{t,x}\nu(dx)}{\left(\beta_t^2 + \int_{\mathbb{R}_0}\gamma^2_{t,x}\nu(dx)\right)^2}dt\right] \\ &\leq 2\mathbb{E}\left[\int_0^T \left\{(h^0_t)^2 + \int_{\mathbb{R}_0}(h^1_{t,z})^2\nu(dz)\right\}dt\right]. \end{aligned}$$

By the same way as the above, we can see $\mathbb{E}\left[\int_0^T\int_{\mathbb{R}_0}\xi_t^2 S_{t-}^2\gamma^2_{t,z}\nu(dz)dt\right]$. Together with (5.3.1), Lemma 5.3.1 follows. □

Consequently, we can conclude the following:

Theorem 5.3.2 *Assume that Assumptions 5.2.1, 5.2.6, and (5.3.1). We have then $\xi^F = \xi$ defined in (5.3.2).*

In the above theorem, a representation of LRM ξ^F is obtained under a mild setting. Since the processes h^0 and h^1 appeared in (5.3.2) are induced by the martingale representation theorem, it is almost impossible to calculate them explicitly, and confirm if (5.3.1) holds. In the rest of this section, we aim to get concrete expressions for h^0 and h^1 by using Malliavin calculus.

5.3.2 Main results

We now calculate h^0 and h^1 by using Theorem 4.5.3. Together with Theorem 5.3.2, we obtain the following:

Theorem 5.3.3 *Under Assumptions 5.2.1, 5.2.6 and 4.5.1, h^0 and h^1 are described as*

$$h_t^0 = \sigma \mathbb{E}_{\mathbb{P}^*}\left[D_{t,0}F - F\left[\int_0^T D_{t,0}u_s dW_s^{\mathbb{P}^*} + \int_0^T \int_{\mathbb{R}_0} \frac{D_{t,0}\theta_{s,x}}{1-\theta_{s,x}} \widetilde{N}^{\mathbb{P}^*}(ds,dx) \right] \Big| \mathcal{F}_{t-} \right], \quad (5.3.3)$$

$$h_{t,z}^1 = \mathbb{E}_{\mathbb{P}^*}[F(H_{t,z}^* - 1) + zH_{t,z}^* D_{t,z}F | \mathcal{F}_{t-}]. \quad (5.3.4)$$

Moreover, LRM ξ^F is given by substituting (5.3.3) and (5.3.4) for h^0 and h^1 in (5.3.2) respectively, if (5.3.1) holds.

Remark 5.3.4
1. *LRM for Lévy markets has been also discussed in Vandaele and Vanmaele [52] without Malliavin calculus. They considered the case where all coefficients in (5.2.1) are deterministic; and studied LRM for unit-linked life insurance contracts.*
2. *Benth et al [9] also concerned a similar issue by using Malliavin calculus. They however studied minimal variance portfolio which is different from LRM, and considered only the case where the underlying asset price process is a martingale.*
3. *Yang et al. [54] derived an explicit representation of LRM for a European call option in the Hull and White model by using the Malliavin calculus in Wiener space. They also give a numerical result of it.*

In order to calculate LRM concretely through Theorem 5.3.3, we need to confirm if all the assumptions in Theorem 5.3.3 are satisfied for a given model. But, it seems to be a hard work. So that, we introduce a simple framework satisfying all the assumptions.

Example 5.3.5 *We consider the case where α, β and γ in (5.2.1) are deterministic functions satisfying the three conditions in Example 5.2.8. Additionally, we assume that*

$$Z_T F \in L^2(\mathbb{P}), \text{ and condition 5 in Assumption 4.5.1.} \quad (5.3.5)$$

Now, we confirm if this model satisfies all the conditions in Theorem 5.3.3. Remark that we discuss this framework in sections 5.4 and 5.5 again for the case where F is a call option or an Asian option.

As seen in Example 5.2.8, Assumption 5.2.1 is satisfied; and Z is a positive square integrable martingale. Thus, together with the above additional condition, Assumption 5.2.6 is satisfied. Since u is bounded and deterministic, condition 1 of Assumption 4.5.1 is satisfied. Since θ is deterministic, the third condition in Example 5.2.8 ensures that condition 3 holds with $\varepsilon \in (0,1)$

independent of $(t,z) \in [0,T] \times \mathbb{R}_0$. *Note that* $|x + \log(1-x)| \leq \frac{1}{2\varepsilon}|x|^2$, *and* $|\log(1-x)| \leq \frac{-\log\varepsilon}{1-\varepsilon}|x|$ *hold for any* $x < 1-\varepsilon$. *Then,* $\int_0^T \int_{\mathbb{R}_0} |\theta_{t,z}|^2 \nu(dz)dt < \infty$ *implies condition 2. As for condition 4, noting that Propositions 4.3.4 and 4.3.3; and Proposition 4.3.6, we can see that* $\log Z_T \in \mathbb{D}^{1,2}$, *and* $D_{t,z}\log Z_T = -\sigma^{-1}u_t \mathbf{1}_{\{0\}}(z) + z^{-1}\log(1-\theta_{t,z})\mathbf{1}_{\mathbb{R}_0}(z)$. *In addition, we have*

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}} \left\{ D_{t,0}\log Z_T \mathbf{1}_{\{0\}}(z) + \frac{e^{zD_{t,z}\log Z_T} - 1}{z}\mathbf{1}_{\mathbb{R}_0}(z) \right\}^2 q(dt,dz) \\ &\quad = \int_0^T u_t^2 dt + \int_0^T \int_{\mathbb{R}_0} \theta_{t,z}^2 \nu(dz)dt < \infty,\end{aligned}$$

from which condition 4 follows. Since $H^* = 1$ *identically,* $F \in \mathbb{D}^{1,2}$ *and* $Z_T \in L^2(\mathbb{P})$, *we have condition 6. It remains to make sure of (5.3.1). Note that* $h^0 = \sigma \mathbb{E}_{\mathbb{P}^*}[D_{t,0}F|\mathcal{F}_{t-}]$, *and* $h^1_{t,z} = \mathbb{E}_{\mathbb{P}^*}[zD_{t,z}F|\mathcal{F}_{t-}]$. *Since* $K_T \in L^\infty$, *we can see that* Z *satisfies the reverse Hölder inequality by Proposition 3.7 of Choulli, Krawczyk and Stricker [12]. We have then* $(\mathbb{E}_{\mathbb{P}^*}[D_{t,0}F|\mathcal{F}_{t-}])^2 \leq C\mathbb{E}[(D_{t,0}F)^2|\mathcal{F}_{t-}]$ *for some* $C > 0$. *By Fubini's theorem, (5.3.1) is satisfied.*

Consequently, all the conditions in Theorem 5.3.3 are satisfied; and ξ^F *is given by*

$$\xi_t^F = \frac{\sigma\beta_t \mathbb{E}_{\mathbb{P}^*}[D_{t,0}F|\mathcal{F}_{t-}] + \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[zD_{t,z}F|\mathcal{F}_{t-}]\gamma_{t,z}\nu(dz)}{S_{t-}\left(\beta_t^2 + \int_{\mathbb{R}_0}\gamma_{t,z}^2\nu(dz)\right)}. \tag{5.3.6}$$

5.4 Call options

In this section, we deal with call options as a common example of contingent claims. The payoff of the call option with strike price $K > 0$ is expressed by $(S_T - K)^+$ where $x^+ = x \vee 0$. First of all, we calculate the Malliavin derivatives of $(F-K)^+$ for $F \in \mathbb{D}^{1,2}$ and $K \in \mathbb{R}$. After that, we shall give an explicit representation of LRM for the deterministic coefficients case discussed in Example 5.3.5.

Regarding $(F-K)^+$ as a functional of F, it is continuous, but not smooth. Thus, we cannot use the chain rule (Propositions 4.2.8 and 4.2.9). Instead, the mollifier approximation is very useful.

Theorem 5.4.1 *For any* $F \in \mathbb{D}^{1,2}$, $K \in \mathbb{R}$ *and* q*-a.e.* $(t,z) \in [0,T] \times \mathbb{R}$, *we have* $(F-K)^+ \in \mathbb{D}^{1,2}$ *and*

$$D_{t,z}(F-K)^+ = \mathbf{1}_{\{F>K\}}D_{t,0}F \cdot \mathbf{1}_{\{0\}}(z) + \frac{(F + zD_{t,z}F - K)^+ - (F-K)^+}{z}\mathbf{1}_{\mathbb{R}_0}(z).$$

Proof. We take a mollifier function φ which is a C^∞-function from $\mathbb{R}$ to $[0,\infty)$ with $\mathrm{supp}(\varphi) \subset [-1,1]$ and $\int_{-\infty}^\infty \varphi(x)dx = 1$. We denote $\varphi_n(x) := n\varphi(nx)$ and $f_n(x) := \int_{-\infty}^\infty (y-K)^+\varphi_n(x-y)dy$ for any $n \geq 1$. Noting that

$$f_n(x) = \int_{-\infty}^\infty \left(x - \frac{y}{n} - K\right)^+ \varphi(y)dy = \int_{-\infty}^{n(x-K)} \left(x - \frac{y}{n} - K\right)\varphi(y)dy,$$

we have $f_n'(x) = \int_{-\infty}^{n(x-K)} \varphi(y)dy$, so that $f_n \in C^1$ and $|f_n'| \leq 1$, that is, f_n is Lipschitz continuous with constant 1. Thus, Proposition 4.2.8 implies that, for any $n \geq 1$, $f_n(F) \in \mathbb{D}^{1,2}$ and

$$D_{t,z}f_n(F) = f_n'(F)D_{t,0}F \cdot \mathbf{1}_{\{0\}}(z) + \frac{f_n(F + zD_{t,z}F) - f_n(F)}{z}\mathbf{1}_{\mathbb{R}_0}(z). \tag{5.4.1}$$

In addition, noting that

$$\begin{aligned} |f_n(x) - (x-K)^+| &= \left| \int_{-1}^{1} \left\{ \left(x - \frac{y}{n} - K \right)^+ - (x-K)^+ \right\} \varphi(y)dy \right| \\ &\leq \frac{1}{n} \int_{-1}^{1} |y|\varphi(y)dy \leq \frac{1}{n} \end{aligned} \tag{5.4.2}$$

for any $x \in \mathbb{R}$, we have $\lim_{n\to\infty} \mathbb{E}[|f_n(F) - (F-K)^+|^2] = 0$. Thus, from the view of Proposition 4.2.6, all we have to do is to make sure that $D_{t,z}f_n(F)$ converges to

$$\mathbf{1}_{\{F>K\}}D_{t,0}F \cdot \mathbf{1}_{\{0\}}(z) + \frac{(F + zD_{t,z}F - K)^+ - (F-K)^+}{z}\mathbf{1}_{\mathbb{R}_0}(z) =: I_\infty$$

in $L^2(q \times \mathbb{P})$ as n tends to ∞.

First of all, we have

$$\lim_{n\to\infty} f_n'(x) = \begin{cases} \int_{-\infty}^{0} \varphi(y)dy & \text{if } x = K, \\ 1 & \text{if } x > K, \\ 0 & \text{if } x < K, \end{cases}$$

from which we obtain $\lim_{n\to\infty} f_n'(F) = \mathbf{1}_{\{F>K\}} + \mathbf{1}_{\{F=K\}} \int_{-\infty}^{0} \varphi(y)dy$. By (5.4.1), (5.4.2) and Lemma 5.4.2 below, we have $\lim_{n\to\infty} D_{t,z}f_n(F) = I_\infty$ in $q \times \mathbb{P}$-a.e., and

$$\begin{aligned} &|D_{t,z}f_n(F) - I_\infty| \\ &\leq |f_n'(F)D_{t,0}F - \mathbf{1}_{\{F>K\}}D_{t,0}F|\mathbf{1}_{\{0\}}(z) \\ &\quad + \left| \frac{f_n(F + zD_{t,z}F) - f_n(F)}{z} - \frac{(F + zD_{t,z}F - K)^+ - (F-K)^+}{z} \right| \mathbf{1}_{\mathbb{R}_0}(z) \\ &\leq 2|D_{t,z}F| \in L^2(q \times \mathbb{P}). \end{aligned}$$

Thus, the dominated convergence theorem provides that $D_{t,z}f_n(F) \to I_\infty$ in $L^2(q \times \mathbb{P})$. □

Lemma 5.4.2 *For any $F \in \mathbb{D}^{1,2}$, we have $\mathbf{1}_{\{F=0\}}D_{t,0}F = 0$, (t,ω)-a.e.*

Proof. *Step 1.* We take the same mollifier function φ as Theorem 5.4.1. Additionally, we assume that $\varphi(0) = 1$. We denote, for any $n \geq 1$, $\varphi_n(x) := \varphi(nx)$ and $\Phi_n(x) := \int_{-\infty}^{x} \varphi_n(y)dy$. Remark that $\Phi_n \in C^1$; and $\Phi_n'(x) = \varphi_n(x)$ is bounded. Proposition 4.2.8 implies

$$D_{t,0}\Phi_n(F) = \varphi_n(F)D_{t,0}F. \tag{5.4.3}$$

Since $\varphi_n(x) \to \mathbf{1}_{\{0\}}(x)\varphi(0) = \mathbf{1}_{\{0\}}(x)$ for any $x \in \mathbb{R}$, we have

$$\lim_{n\to\infty} D_{t,0}\Phi_n(F) = \mathbf{1}_{\{F=0\}} D_{t,0}F. \tag{5.4.4}$$

Step 2. Recall that any function $u \in L^2(q \times \mathbb{P})$ has a chaotic representation

$$u(t,z) = \sum_{n=0}^{\infty} I_n(h_n(\cdot,(t,z))),$$

where $h_n \in L^2_{T,q,n+1}$ is symmetric in the first n pairs of variables. Denoting by $\hat{h}_n$ the symmetrization of h_n with respect to all $n+1$ pairs of variables, we define

$$\mathrm{Dom}_\delta := \left\{ u \in L^2(q \times \mathbb{P}) \middle| \sum_{n=0}^{\infty} (n+1)! \|\hat{h}_n\|^2_{L^2_{T,q,n+1}} < \infty \right\}.$$

We shall show that Dom_δ is dense in $L^2(q \times \mathbb{P})$. Now, we prepare a subclass of Dom_δ as

$$\mathrm{Dom}_f := \left\{ u \in L^2(q \times \mathbb{P}) \middle| u(t,z) = \sum_{n=0}^{N} I_n(h_n(\cdot,(t,z))) \text{ for some } N \geq 1 \right\}.$$

Taking a $u \in L^2(q \times \mathbb{P})$ with $u(t,z) = \sum_{n=0}^{\infty} I_n(h_n(\cdot,(t,z)))$ arbitrarily; and denoting $u_N(t,z) := \sum_{n=0}^{N} I_n(h_n(\cdot,(t,z))) \in \mathrm{Dom}_f$ for any $N \geq 1$, we have $u_N \to u$ in $L^2(q \times \mathbb{P})$. Thus, Dom_f is dense in $L^2(q \times \mathbb{P})$. So is Dom_δ.

Step 3. By the dense property of Dom_δ, we have only to see

$$\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} \mathbf{1}_{\{F=0\}} D_{t,0}F \cdot \mathbf{1}_{\{0\}}(z) u(t,z) q(dt,dz)\right] = 0 \tag{5.4.5}$$

for any $u \in \mathrm{Dom}_\delta$. Fix $u \in \mathrm{Dom}_\delta$ arbitrarily. By (5.4.4), we have

$$\mathbb{E}\left[\int_0^T \mathbf{1}_{\{F=0\}} D_{t,0}F \cdot u(t,0)dt\right] = \mathbb{E}\left[\int_0^T \lim_{n\to\infty} D_{t,0}\Phi_n(F) \cdot u(t,0)dt\right]. \tag{5.4.6}$$

Since we can find a $C_\varphi > 0$ such that $\varphi \leq C_\varphi$, (5.4.3) implies

$$|D_{t,0}\Phi_n(F)| \leq |\varphi_n(F)||D_{t,0}F| \leq C_\varphi |D_{t,0}F|.$$

In addition, we have

$$\mathbb{E}\left[\int_0^T |D_{t,0}F \cdot u(t,0)|dt\right] \leq \sqrt{\mathbb{E}\left[\int_0^T |D_{t,0}F|^2 dt\right]} \sqrt{\mathbb{E}\left[\int_0^T |u(t,0)|^2 dt\right]} < \infty.$$

Thus, the dominated convergence theorem yields

$$\mathbb{E}\left[\int_0^T \lim_{n\to\infty} D_{t,0}\Phi_n(F) \cdot u(t,0)dt\right] = \lim_{n\to\infty} \mathbb{E}\left[\int_0^T D_{t,0}\Phi_n(F) \cdot u(t,0)dt\right]. \tag{5.4.7}$$

Next, by the duality formula (Proposition 4.3.2), there exists a constant $C > 0$ such that

$$\left|\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} D_{t,z}\Phi_n(F)\cdot u(t,z)q(dt,dz)\right]\right| \le C\|\Phi_n(F)\|_{L^2(\mathbb{P})} \le C\frac{1}{n},$$

which means

$$\mathbb{E}\left[\int_0^T D_{t,0}\Phi_n(F)\cdot u(t,0)dt\right] \to 0 \tag{5.4.8}$$

as $n \to \infty$. Consequently, (5.4.6), (5.4.7) and (5.4.8) imply (5.4.5). □

5.4.1 The deterministic coefficients case

Throughout this subsection, we consider the case where α, β and γ in (5.2.1) are deterministic functions satisfying the three conditions in Example 5.2.8. Additionally, we assume the following condition:

$$\int_{\mathbb{R}_0}\{\gamma_{t,z}^4 + |\log(1+\gamma_{t,z})|^2\}\nu(dz) < C \text{ for some } C > 0. \tag{5.4.9}$$

We aim to obtain a concrete representation of LRM for the call option $(S_T - K)^+$. As seen in Example 5.3.5, this model satisfies all the conditions in Theorem 5.3.3, if (5.3.5) is satisfied. First of all, we calculate the Malliavin derivatives of S_T.

Proposition 5.4.3 *We have $S_T \in \mathbb{D}^{1,2}$; and*

$$D_{t,z}S_T = \frac{S_T\beta_t}{\sigma}\mathbf{1}_{\{0\}}(z) + \frac{S_T\gamma_{t,z}}{z}\mathbf{1}_{\mathbb{R}_0}(z) \tag{5.4.10}$$

for q-a.e. $(t,z) \in [0,T]\times\mathbb{R}$.

Proof. Noting that

$$\begin{aligned}\log(S_T/S_0) = &\int_0^T\left[\alpha_t - \frac{1}{2}\beta_t^2 + \int_{\mathbb{R}_0}\{\log(1+\gamma_{t,z}) - \gamma_{t,z}\}\,\nu(dz)\right]dt \\ &+ \int_0^T \beta_t dW_t + \int_0^T\int_{\mathbb{R}_0}\log(1+\gamma_{t,z})\widetilde{N}(dt,dz),\end{aligned}$$

we have $\log(S_T/S_0) \in \mathbb{D}^{1,2}$ and $D_{t,z}\log(S_T/S_0) = \frac{\beta_t}{\sigma}\mathbf{1}_{\{0\}}(z) + \frac{\log(1+\gamma_{t,z})}{z}\mathbf{1}_{\mathbb{R}_0}(z)$ for any $(t,z) \in [0,T]\times\mathbb{R}$ by (5.4.9) and Proposition 4.3.3. Setting $F := \log(S_T/S_0)$ and $f(x) := S_0e^x$, we have $S_T = f(F)$. Thus, we have $f'(F)D_{t,0}F = S_T\frac{\beta_t}{\sigma}$ for any $t \in [0,T]$; and

$$\frac{f(F+zD_{t,z}F) - f(F)}{z} = S_T\frac{\exp\{zD_{t,z}F\} - 1}{z} = \frac{S_T\gamma_{t,z}}{z}$$

for any $(t,z) \in [0,T]\times\mathbb{R}_0$. Hence, Proposition 4.2.9 implies $S_T \in \mathbb{D}^{1,2}$ and (5.4.10). □

Remark 5.4.4 *A similar argument with Proposition 5.4.3, together with Example 5.3.5, yields*

$$D_{t,z}Z_T = -Z_T\left(\frac{u_t}{\sigma}\mathbf{1}_{\{0\}}(z) + \frac{\theta_{t,z}}{z}\mathbf{1}_{\mathbb{R}_0}(z)\right).$$

Now, we confirm condition (5.3.5).

Lemma 5.4.5 *Condition (5.3.5) in Example 5.3.5 is satisfied.*

Proof. By simple calculations, we have

$$d(Z_tS_t) = S_{t-}Z_{t-}\left\{(\beta_t - u_t)dW_t + \int_{\mathbb{R}_0}(\gamma_{t,z} - \theta_{t,z} - \gamma_{t,z}\theta_{t,z})\widetilde{N}(dt,dz)\right\},$$

which implies $Z_TS_T \in L^2(\mathbb{P})$ by Theorem 117 of Situ [48]. Therefore, $Z_T(S_T - K)^+ \in L^2(\mathbb{P})$ holds.

Since Theorem 5.4.1 and Proposition 5.4.3 imply that $(S_T - K)^+ \in \mathbb{D}^{1,2}$, and

$$D_{t,z}(S_T - K)^+ = \mathbf{1}_{\{S_T>K\}}\frac{S_T\beta_t}{\sigma}\cdot\mathbf{1}_{\{0\}}(z) + \frac{(S_T(1+\gamma_{t,z}) - K)^+ - (S_T - K)^+}{z}\mathbf{1}_{\mathbb{R}_0}(z), \tag{5.4.11}$$

we have

$$\left\|Z_TD_{t,z}(S_T - K)^+\right\|^2_{L^2(q\times\mathbb{P})} \le \mathbb{E}[Z_T^2S_T^2]\left(\int_0^T\beta_t^2dt + \int_0^T\int_{\mathbb{R}_0}\gamma_{t,z}^2\nu(dz)dt\right) < \infty,$$

and

$$\left\|(S_T - K)^+D_{t,z}Z_T\right\|^2_{L^2(q\times\mathbb{P})} \le \mathbb{E}[S_T^2Z_T^2]\left(\int_0^T u_t^2dt + \int_0^T\int_{\mathbb{R}_0}\theta_{t,z}^2\nu(dz)dt\right) < \infty.$$

In addition, there is a $C > 0$ such that

$$\begin{aligned}&\mathbb{E}\left[\int_0^T\int_{\mathbb{R}}|zD_{t,z}(S_T - K)^+D_{t,z}Z_T|^2q(dt,dz)\right]\\&\le \mathbb{E}[Z_T^2S_T^2]\left(\int_0^T\int_{\mathbb{R}_0}\gamma_{t,z}^2\theta_{t,z}^2\nu(dz)dt\right) \le C\mathbb{E}[Z_T^2S_T^2]\left(\int_0^T\int_{\mathbb{R}_0}\gamma_{t,z}^4\nu(dz)dt\right),\end{aligned}$$

from which condition 5 in Example 4.5.1 follows by (5.4.9). This completes the proof. □

Next, by using the above proposition and lemma, we can calculate an explicit representation of LRM for call options as follows:

Proposition 5.4.6 *For any $K > 0$ and $t \in [0, T]$, we have*

$$\begin{aligned}\xi_t^{(S_T-K)^+} = &\frac{1}{S_{t-}\left(\beta_t^2 + \int_{\mathbb{R}_0}\gamma_{t,z}^2\nu(dz)\right)}\Bigg\{\beta_t^2\mathbb{E}_{\mathbb{P}^*}[\mathbf{1}_{\{S_T>K\}}S_T|\mathcal{F}_{t-}]\\&+\int_{\mathbb{R}_0}\mathbb{E}_{\mathbb{P}^*}[(S_T(1+\gamma_{t,z}) - K)^+ - (S_T - K)^+|\mathcal{F}_{t-}]\gamma_{t,z}\nu(dz)\Bigg\}.\end{aligned} \tag{5.4.12}$$

Proof. From the view of Lemma 5.4.5, (5.4.12) is given by (5.3.6) and (5.4.11). □

Remark 5.4.7 *By using this result, we also study numerical analysis of LRM for exponential Lévy models in Arai, Imai and Suzuki [4].*

5.5 Asian Options

In this section, we study Asian options, whose payoff is depending on $\frac{1}{T}\int_0^T S_s ds$. First of all, Proposition 4.3.4 implies the following proposition:

Proposition 5.5.1 *Besides Assumption 5.2.1, we assume the following two conditions:*

1. $S_s \in \mathbb{D}^{1,2}$ *for a.e.* $s \in [0,T]$.
2. $\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}}\int_{[0,T]} |D_{t,z}S_s|^2 ds q(dt,dz)\right] < \infty.$

We have then $\frac{1}{T}\int_0^T S_s ds \in \mathbb{D}^{1,2}$ *and* $D_{t,z}\frac{1}{T}\int_0^T S_s ds = \frac{1}{T}\int_0^T D_{t,z}S_s ds$ *for q-a.e.* $(t,z) \in [0,T]\times\mathbb{R}$.

Next, we calculate Malliavin derivatives and LRM of Asian options for the same setting as subsection 5.4.1.

Proposition 5.5.2 *When* α, β *and* γ *are deterministic functions satisfying the three conditions in Example 5.2.8 and (5.4.9), we have* $\frac{1}{T}\int_0^T S_s ds \in \mathbb{D}^{1,2}$ *and*

$$D_{t,z}\frac{1}{T}\int_0^T S_s ds = \frac{1}{T}\left\{\frac{\beta_t}{\sigma}\mathbf{1}_{\{0\}}(z) + \frac{\gamma_{t,z}}{z}\mathbf{1}_{\mathbb{R}_0}(z)\right\}\int_t^T S_s ds$$

for q-a.e. $(t,z) \in [0,T]\times\mathbb{R}$.

Proof. By the same way as Proposition 5.4.3, we can see that condition 1 in Proposition 5.5.1 and

$$D_{t,z}S_s = S_s\mathbf{1}_{[0,s]}(t)\left\{\frac{\beta_t}{\sigma}\mathbf{1}_{\{0\}}(z) + \frac{\gamma_{t,z}}{z}\mathbf{1}_{\mathbb{R}_0}(z)\right\}$$

for q-a.e. $(t,z) \in [0,T]\times\mathbb{R}$ and any $s \in [0,T]$. As for condition 2, we have the following:

$$\begin{aligned}
&\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}}\int_0^T |D_{t,z}S_s|^2 ds q(dt,dz)\right] \\
&\leq T\mathbb{E}\left[\sup_{s\in[0,T]} S_s^2\right]\left(\int_0^T \beta_t^2 dt + \int_{[0,T]\times\mathbb{R}_0}\gamma_{t,z}^2\nu(dz)dt\right) < \infty.
\end{aligned}$$

□

We illustrate LRM for Asian options with payoff $(\frac{1}{T}\int_0^T S_s ds - K)^+$.

Proposition 5.5.3 *Under the same setting as Proposition 5.5.2, we have, for any $K > 0$ and $t \in [0, T]$,*

$$\zeta_t^{(V_0-K)^+} = \frac{1}{S_{t-}\left(\beta_t^2 + \int_{\mathbb{R}_0} \gamma_{t,z}^2 \nu(dz)\right)} \left\{ \beta_t^2 \mathbb{E}_{\mathbb{P}^*}[\mathbf{1}_{\{V_0>K\}} V_t | \mathcal{F}_{t-}] \right.$$
$$\left. + \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}\left[(V_0 + \gamma_{t,z} V_t - K)^+ - (V_0 - K)^+ | \mathcal{F}_{t-}\right] \gamma_{t,z} \nu(dz) \right\},$$

where $V_t = \frac{1}{T}\int_t^T S_s ds$ for $t \in [0, T]$.

Proof. Theorem 5.4.1 and Propositions 5.5.2 imply that

$$D_{t,z}(V_0 - K)^+ = \mathbf{1}_{\{V_0>K\}} \frac{\beta_t V_t}{\sigma} \mathbf{1}_{\{0\}}(z) + \frac{(V_0 + \gamma_{t,z} V_t - K)^+ - (V_0 - K)^+}{z} \mathbf{1}_{\mathbb{R}_0}(z).$$

Thus, this proposition is concluded by (5.3.6). □

5.6 Lookback Options

We focus on lookback options, that is, options whose payoff depends on the running maximum of the asset price process $M^S := \sup_{t\in[0,T]} S_t$. We treat only the exponential Lévy case in this section.

5.6.1 Malliavin derivatives of running maximum

First of all, we calculate Malliavin derivatives of the running maximum over $[0, T]$ of the following Lévy process: $L_t = \mu t + X_t$, where X is the underlying Lévy process defined in (4.2.2), and $\mu \in \mathbb{R}$. Note that $L_t \in \mathbb{D}^{1,2}$ for any $t \in [0, T]$. Before stating the main theorem, we need some preparations.

Lemma 5.6.1 *Let $F_1, F_2, \cdots \in \mathbb{D}^{1,2}$. We have then, for any $n \geq 1$, $M_n := \max_{1\leq k\leq n} F_k \in \mathbb{D}^{1,2}$ and*

$$D_{t,z} M_n = \sum_{k=1}^{n} \mathbf{1}_{A_{n,k}} D_{t,0} F_k \cdot \mathbf{1}_{\{0\}}(z) + \frac{\max_{1\leq k\leq n}(F_k + z D_{t,z} F_k) - M_n}{z} \mathbf{1}_{\mathbb{R}_0}(z), \qquad (5.6.1)$$

where $A_{n,1} = \{M_n = F_1\}$ and $A_{n,k} = \{M_n \neq F_1, \cdots, M_n \neq F_{k-1}, M_n = F_k\}$ for $2 \leq k \leq n$.

Proof. Remark that $M_2 = F_1 \vee F_2 = (F_2 - F_1)^+ + F_1 \in \mathbb{D}^{1,2}$ by Theorem 5.4.1; and $M_n = F_n \vee M_{n-1}$. We have then $M_n \in \mathbb{D}^{1,2}$ for any $n \geq 1$.

Next, we calculate $D_{t,0}M_n$. Theorem 5.4.1 implies

$$\begin{aligned}D_{t,0}M_n &= D_{t,0}(F_n \vee M_{n-1}) = D_{t,0}(F_n - M_{n-1})^+ + D_{t,0}M_{n-1} \\ &= \mathbf{1}_{\{F_n > M_{n-1}\}} D_{t,0}(F_n - M_{n-1}) + D_{t,0}M_{n-1} \\ &= \mathbf{1}_{\{F_n > M_{n-1}\}} D_{t,0}F_n + \mathbf{1}_{\{F_n \le M_{n-1}\}} D_{t,0}M_{n-1} \\ &= \mathbf{1}_{A_{n,n}} D_{t,0}F_n + \mathbf{1}_{\{M_n = M_{n-1}\}} D_{t,0}M_{n-1}.\end{aligned}$$

Iterating this calculation shows

$$\begin{aligned}D_{t,0}M_n &= \mathbf{1}_{A_{n,n}} D_{t,0}F_n \\ &\quad + \mathbf{1}_{\{M_n = M_{n-1}\}} \left\{ \mathbf{1}_{A_{n-1,n-1}} D_{t,0}F_{n-1} + \mathbf{1}_{\{M_{n-1} = M_{n-2}\}} D_{t,0}M_{n-2} \right\} \\ &= \mathbf{1}_{A_{n,n}} D_{t,0}F_n + \mathbf{1}_{A_{n,n-1}} D_{t,0}F_{n-1} + \mathbf{1}_{\{M_n = M_{n-2}\}} D_{t,0}M_{n-2} \\ &= \cdots = \sum_{k=1}^{n} \mathbf{1}_{A_{n,k}} D_{t,0}F_k. \end{aligned} \tag{5.6.2}$$

For the case where $z \neq 0$, we have

$$\begin{aligned}D_{t,z}M_n &= D_{t,z}(F_n - M_{n-1})^+ + D_{t,z}M_{n-1} \\ &= \frac{(F_n - M_{n-1} + zD_{t,z}(F_n - M_{n-1}))^+ - (F_n - M_{n-1})^+}{z} + D_{t,z}M_{n-1} \\ &= \frac{1}{z}\Big[(F_n - M_{n-1} + zD_{t,z}(F_n - M_{n-1}))^+ + M_{n-1} + zD_{t,z}M_{n-1} \\ &\quad - \{(F_n - M_{n-1})^+ + M_{n-1}\} \Big] \\ &= \frac{(F_n + zD_{t,z}F_n) \vee (M_{n-1} + zD_{t,z}M_{n-1}) - F_n \vee M_{n-1}}{z},\end{aligned}$$

that is, $M_n + zD_{t,z}M_n = (F_n + zD_{t,z}F_n) \vee (M_{n-1} + zD_{t,z}M_{n-1})$. Thus, we have

$$\begin{aligned}M_n + zD_{t,z}M_n &= (F_n + zD_{t,z}F_n) \vee (M_{n-1} + zD_{t,z}M_{n-1}) \\ &= (F_n + zD_{t,z}F_n) \vee (F_{n-1} + zD_{t,z}F_{n-1}) \vee (M_{n-2} + zD_{t,z}M_{n-2}) \\ &= \cdots = \max_{1 \le k \le n} (F_k + zD_{t,z}F_k),\end{aligned}$$

which means

$$D_{t,z}M_n = \frac{\max_{1 \le k \le n}(F_k + zD_{t,z}F_k) - M_n}{z}. \tag{5.6.3}$$

By (5.6.2) and (5.6.3), we obtain (5.6.1). □

We need to show more two lemmas. We take a countable dense subset $\mathcal{U} := \{u_1, u_2, \cdots\} \subset [0,T]$ with $T \in \mathcal{U}$.

Lemma 5.6.2 *Let $\{Y_t\}_{t \in [0,T]}$ be an RCLL process. Denoting $M_n^Y := \max_{1 \le k \le n} Y_{u_k}$ for any $n \ge 1$; and $M^Y := \sup_{t \in [0,T]} Y_t$, we have $M_n^Y \to M^Y$ as $n \to \infty$.*

Proof. Since $M_n^Y \le M^Y$ for any $n \ge 1$, it suffices to show that $\mathbb{P}(\lim_{n\to\infty} M_n^Y < M^Y) = 0$. Now, suppose that $\mathbb{P}\left(\lim_{n\to\infty} M_n^Y < M^Y\right) > 0$. Denoting $A_k := \{M^Y - \lim_{n\to\infty} M_n^Y \ge 1/k\}$ for $k \ge 1$, we have $0 < \mathbb{P}(\lim_{n\to\infty} M_n^Y < M^Y) = \mathbb{P}\left(\cup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} \mathbb{P}(A_k)$. Thus, $\mathbb{P}(A_k) > 0$ holds for any sufficiently large k. Now, fix such a k arbitrarily. Note that there exists a $[0,T)$-valued random time ζ such that $Y_\zeta \ge M^Y - \frac{1}{2k}$ on A_k, since we can find a $[0,T]$-valued random time $\hat{\zeta}$ such that $Y_{\hat{\zeta}} \ge M^Y - \frac{1}{2k}$ a.s., but $Y_T \le M^Y - \frac{1}{k}$ on A_k because $T \in \mathcal{U}$. By the dense property of $\mathcal{U}$ and the RCLL property of Y, we can find a $\mathcal{U}$-valued random time η such that $Y_\eta > M^Y - \frac{1}{k}$ on A_k. This is a contradiction to the definition of A_k. □

To see Lemma 5.6.3 below, we denote $M_n^L := \max_{1\le k\le n} L_{u_k}$ for any $n \ge 1$, $M^L := \sup_{t\in[0,T]} L_t$, and $\tau := \inf\{t \in [0,T] | L_t \vee L_{t-} = M^L\}$. Note that $M^L = \sup_{t\in[0,T]}(L_t \vee L_{t-}) = L_\tau \vee L_{\tau-}$; and τ is a unique random time satisfying $M^L = L_\tau \vee L_{\tau-}$ by Lemma 49.4 of Sato [41].

Lemma 5.6.3 $\mathbb{P}(\tau = t) = 0$ *for any* $t \in [0,T]$.

Proof. Taking a $t \in [0,T)$ arbitrarily, we have

$$\mathbb{P}\left(\limsup_{s\downarrow 0} \frac{L_{t+s} - L_t}{s} = +\infty\right) = 1$$

by Theorem 47.1 and Proposition 10.7 of Sato [41]. Thus, $\mathbb{P}(L_{t+s} \le L_t \text{ for any } s \in (0, T-t]) = 0$ holds, from which $\mathbb{P}(L_t = M^L) = 0$ follows. On the other hand, $\mathbb{P}(L_{t-} = L_t) = 1$ by Proposition I.7 of Bertoin [10]. As a result, we obtain $\mathbb{P}(\tau = t) = 0$ for any $t \in [0,T)$. As for the case of $t = T$, Theorem 47.1 of Sato [41] together with Lemma II.2 of Bertoin [10] provides

$$\mathbb{P}\left(\limsup_{s\downarrow 0} \frac{L_{(T-s)-} - L_T}{s} = +\infty\right) = \mathbb{P}\left(\liminf_{s\downarrow 0} \frac{L_s}{s} = -\infty\right) = 1,$$

which implies $\mathbb{P}(L_s \le L_T \text{ for any } s \in [0,T)) = 0$. By the same argument as the above, $\mathbb{P}(\tau = T) = 0$ follows. □

At last, we introduce the main theorem of this subsection.

Theorem 5.6.4 $M^L \in \mathbb{D}^{1,2}$ *and*

$$D_{t,z}M^L = \mathbf{1}_{\{\tau\ge t\}}\mathbf{1}_{\{0\}}(z) + \frac{\sup_{s\in[0,T]}(L_s + z\mathbf{1}_{\{t\le s\}}) - M^L}{z}\mathbf{1}_{\mathbb{R}_0}(z). \tag{5.6.4}$$

Proof. Noting that $M^L \in L^2(\mathbb{P})$ by the square integrability of X, $M_n^L \in \mathbb{D}^{1,2}$ for any $n \ge 1$ by Lemma 5.6.1; and $M_n^L \to M^L$ in $L^2(\mathbb{P})$ by Lemma 5.6.2 (because, for any n, $|M_n^L| \le M^L$, hence, the sequence (M_n^L) is uniformly integrable), we have only to see that $D_{t,z}M_n^L$ converges to the RHS of (5.6.4) in $L^2(q \times \mathbb{P})$ in view of Proposition 4.2.6.

Step 1. Firstly, we consider the case of $z \neq 0$. Lemma 5.6.1 implies

$$D_{t,z}M_n^L = \frac{\max_{1\le k\le n}(L_{u_k} + zD_{t,z}L_{u_k}) - M_n^L}{z}.$$

Remark that $D_{t,z}L_s = \mathbf{1}_{\{s\ge t\}}$, which is RCLL on s. Thus, Lemma 5.6.2 yields

$$\lim_{n\to\infty} D_{t,z}M_n^L = \frac{\sup_{s\in[0,T]}(L_s + zD_{t,z}L_s) - M^L}{z}. \tag{5.6.5}$$

Moreover, noting that $|\max_{1\le k\le n}(a_k+b_k) - \max_{1\le k\le n} a_k| \le \max_{1\le k\le n}|b_k|$ for any $\{a_k\}_{1\le k\le n}, \{b_k\}_{1\le k\le n} \subset \mathbb{R}$, we obtain

$$\left|\max_{1\le k\le n}(L_{u_k} + zD_{t,z}L_{u_k}) - M_n^L\right| \le \max_{1\le k\le n}|zD_{t,z}L_{u_k}|.$$

Thus, for any $z \in \mathbb{R}_0$,

$$\begin{aligned}
&\left|D_{t,z}M_n^L - \frac{\sup_{u\in[0,T]}(L_u + zD_{t,z}L_u) - M^L}{z}\right|^2 \\
&\le 2\left\{|D_{t,z}M_n^L|^2 + \frac{|\sup_{s\in[0,T]}(L_s + zD_{t,z}L_s) - M^L|^2}{|z|^2}\right\} \\
&\le \frac{2}{|z|^2}\left\{\left|\max_{1\le k\le n}(L_{u_k} + zD_{t,z}L_{u_k}) - M_n^L\right|^2 + \left|\sup_{s\in[0,T]}(L_s + zD_{t,z}L_s) - M^L\right|^2\right\} \\
&\le 2\left\{\max_{1\le k\le n}|D_{t,z}L_{u_k}|^2 + \sup_{s\in[0,T]}|D_{t,z}L_s|^2\right\} \le 4\sup_{s\in[0,T]}|D_{t,z}L_s|^2 = 4.
\end{aligned}$$

The dominated convergence theorem implies that the convergence in (5.6.5) also holds in $L^2(q\times\mathbb{P})$.

Step 2. Next, we see that $D_{t,0}M_n^L \cdot \mathbf{1}_{\{0\}}(z)$ converges to $\mathbf{1}_{\{\tau\ge t\}}\mathbf{1}_{\{0\}}(z)$ in $L^2(q\times\mathbb{P})$. Similarly with Lemma 5.6.1, we denote $A_{n,1}^L = \{M_n^L = L_{u_1}\}$, and $A_{n,k}^L = \{M_n^L \neq L_{u_1}, \cdots, M_n^L \neq L_{u_{k-1}}, M_n^L = L_{u_k}\}$ for $2\le k\le n$. In addition, defining $\tau_n := \sum_{k=1}^n u_k \mathbf{1}_{A_{n,k}^L}$ for any $n\ge 1$, we have

$$D_{t,0}M_n^L = \sum_{k=1}^n \mathbf{1}_{A_{n,k}^L} D_{t,0}L_{u_k} = \sum_{k=1}^n \mathbf{1}_{A_{n,k}^L}\mathbf{1}_{\{u_k\ge t\}} = \mathbf{1}_{\{\tau_n\ge t\}}$$

by Lemma 5.6.1. Recall that $\sup_{s\in[t,T]}(L_s\vee L_{s-}) < L_\tau \vee L_{\tau-}$ on $\{\tau < t\}$ by Lemma 49.4 of Sato [41]. Then, on $\{\tau<t\}$, we can find a $k\in\mathbb{N}$ such that $L_{u_k} > \sup_{s\in[t,T]}(L_s\vee L_{s-})$. Note that k depends on ω. As a result, $\tau_n < t$ holds for any $n\ge k$. Similarly, we can see that, on $\{\tau > t\}$, we have $\tau_n > t$ for any sufficiently large n. Since $\mathbb{P}(\tau = t) = 0$ by Lemma 5.6.3, we can conclude that $\lim_{n\to\infty}\mathbf{1}_{\{\tau_n\ge t\}} = \mathbf{1}_{\{\tau\ge t\}}$ a.s., from which Theorem 5.6.4 follows. □

5.6.2 LRM for lookback options

We consider the case where S_t is given as an exponential Lévy process $S_t = S_0 \exp\{L_t\}$, where $S_0 > 0$; and denote $M^S := \sup_{t\in[0,T]} S_t$. In this subsection, we calculate Malliavin derivatives and LRM of lookback options whose payoffs are given as $M^S - S_T$ and $(M^S - K)^+$ for $K > 0$. Here we assume that $\int_{\mathbb{R}_0} \left\{z^2 + (e^z - 1)^4\right\} \nu(dz) < \infty$; and there exists an $\varepsilon \in (0,1)$ such that

$$\frac{\left\{\mu + \frac{\sigma^2}{2} + \int_{\mathbb{R}_0}(x - e^x + 1)\nu(dx)\right\}(e^z - 1)}{\sigma^2 + \int_{\mathbb{R}_0}(e^x - 1)^2\nu(dx)} < 1 - \varepsilon$$

for ν-a.e. $z \in \mathbb{R}_0$. These conditions are corresponding to (5.4.9) and condition 3 in Example 5.2.8, respectively. Note that the other two conditions in Example 5.2.8 are also satisfied. In addition, $\int_{\mathbb{R}_0}(z - e^z + 1)\nu(dz)$ is well-defined since $e^z - 1 - z \le (e-1)z^2$ for any $z \in [-1,1]$. The following lemma is also given in a similar way with subsection 5.4.1.

Lemma 5.6.5 *(1) We have $M^S \in \mathbb{D}^{1,2}$; and*

$$D_{t,z}M^S = M^S \mathbf{1}_{\{\tau \ge t\}} \mathbf{1}_{\{0\}}(z) + \frac{\sup_{s\in[0,T]}\left(S_s e^{z\mathbf{1}_{\{t\le s\}}}\right) - M^S}{z} \mathbf{1}_{\mathbb{R}_0}(z).$$

(2) Condition (5.3.5) holds for both $M^S - S_T$ and $(M^S - K)^+$.

Proof. (1) Proposition 4.2.9, together with Theorem 5.6.4 and $\int_1^\infty (e^z - 1)^4\nu(dz) < \infty$, implies that $M^S \in \mathbb{D}^{1,2}$, $D_{t,0}M^S = S_0 D_{t,0} e^{M^L} = S_0 e^{M^L} D_{t,0} M^L = M^S \mathbf{1}_{\{\tau \ge t\}}$; and, for $z \in \mathbb{R}_0$,

$$\begin{aligned} D_{t,z}M^S &= S_0 D_{t,z} e^{M^L} = S_0 \frac{\exp\{M^L + z D_{t,z} M^L\} - e^{M^L}}{z} \\ &= S_0 \frac{\exp\left\{\sup_{s\in[0,T]}\left(L_s + z\mathbf{1}_{\{t\le s\}}\right)\right\} - e^{M^L}}{z} \\ &= \frac{\sup_{s\in[0,T]}\left(S_s e^{z\mathbf{1}_{\{t\le s\}}}\right) - M^S}{z}. \end{aligned}$$

(2) We can see this assertion by Lemma 5.4.5. □

Now, we calculate Malliavin derivatives and LRM for lookback options by using Lemma 5.6.5, Theorem 5.4.1 and (5.3.6).

Proposition 5.6.6 *(1)*

$$D_{t,z}(M^S - S_T) = (M^S \mathbf{1}_{\{\tau \geq t\}} - S_T)\mathbf{1}_{\{0\}}(z) + \left(\frac{\sup_{s\in[0,T]}\left(S_s e^{z\mathbf{1}_{\{t\leq s\}}}\right) - M^S}{z} - S_T \frac{e^z - 1}{z} \right) \mathbf{1}_{\mathbb{R}_0}(z).$$

(2) For any $K > 0$, we have

$$D_{t,z}(M^S - K)^+ = M^S \mathbf{1}_{\{M^L > \log(K/S_0)\}} \mathbf{1}_{\{\tau\geq t\}} \mathbf{1}_{\{0\}}(z) + \frac{\left(\sup_{s\in[0,T]}\left(S_s e^{z\mathbf{1}_{\{t\leq s\}}}\right) - K\right)^+ - (M^S - K)^+}{z} \mathbf{1}_{\mathbb{R}_0}(z).$$

Corollary 5.6.7 *For any $K > 0$ and $t \in [0, T]$, we have*

$$\xi_t^{M^S - S_T} = \frac{1}{CS_{t-}} \Bigg\{ \sigma^2 \mathbb{E}_{\mathbb{P}^*}[M^S \mathbf{1}_{\{\tau\geq t\}} - S_T | \mathcal{F}_{t-}] + \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}\left[\sup_{u\in[0,T]} \left(S_u e^{z\mathbf{1}_{\{t\leq u\}}}\right) - M^S - S_T \gamma_z | \mathcal{F}_{t-} \right] \gamma_z \nu(dz) \Bigg\},$$

and

$$\xi_t^{(M^S-K)^+} = \frac{1}{CS_{t-}} \Bigg\{ \sigma^2 \mathbb{E}_{\mathbb{P}^*}[M^S \mathbf{1}_{\{M^L > \log(K/S_0)\}} \mathbf{1}_{\{\tau\geq t\}} | \mathcal{F}_{t-}] + \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}\left[\left(\sup_{u\in[0,T]} \left(S_u e^{z\mathbf{1}_{\{t\leq u\}}}\right) - K \right)^+ - (M^S - K)^+ | \mathcal{F}_{t-} \right] \gamma_z \nu(dz) \Bigg\},$$

where $\gamma_z := e^z - 1$ and $C := \left(\sigma^2 + \int_{\mathbb{R}_0} \gamma_z^2 \nu(dz)\right)$.

Remark 5.6.8 *There are lookback options whose payoff is described by the running minimum of the asset price process, instead of the running maximum. Thus, we should mention about how to calculate Malliavin derivatives for the running minimum of exponential Lévy processes S.*

We denote $m^Y := \inf_{t\in[0,T]} Y_t$ for any stochastic process Y; and $S'_t := 1/S_t = S_0^{-1} e^{-L_t}$. Since S' is also an exponential Lévy process, we can calculate $M^{S'}$ through Theorem 5.6.4. Noting that $M^{S'} \geq S_0^{-1} > 0$, we take a C^1-function f on $\mathbb{R}$ such that $f(x) = 1/x$ if $x \geq S_0^{-1}$. Then, by $m^S = 1/M^{S'}$ and Proposition 4.2.9, we have

$$D_{t,z} m^S = D_{t,z} \frac{1}{M^{S'}} = -\frac{1}{\left(M^{S'}\right)^2} D_{t,z} M^{S'}.$$

Remark that we can calculate $D_{t,z}(S_T - m^S)$ and $D_{t,z}(m^S - K)^+$ by the same way as Proposition 5.6.6.

5.7 Concluding remarks

Throughout this thesis, we consider an incomplete financial market model whose asset price process is given as a solution to the SDE (5.2.1). Under some assumptions, we obtain representation results (Theorem 5.3.3 and Example 5.3.5) of LRM by using Malliavin calculus for Lévy processes based on the canonical Lévy space framework. So that, representations of LRM given in this thesis include Malliavin derivatives of the claim to be hedged.

As typical examples of claims, we treat call options, Asian options and lookback options. As for call options, we formulate their Malliavin derivatives in a general form; and calculate their LRM explicitly for the case where the coefficients of the SDE are deterministic. Next, we illustrate how to calculate Malliavin derivatives of Asian options; and give expressions of their LRM for the deterministic coefficients case. Thirdly, we study lookback options for the exponential Lévy case.

As said above, we calculate LRM for only the deterministic coefficients case. It is because Malliavin derivatives of deterministic functions are given by 0, thereby we can comparatively easily make sure of Assumption 4.5.1 under some mild conditions as seen in subsection 5.4.1. Besides, LRM is expressed simply from the view of Example 5.3.5. On the other hand, in the random coefficients case, we need very complicated calculations to confirm if Assumption 4.5.1 holds. Furthermore, we need to calculate exactly H^* and Malliavin derivatives of u and θ. That's why, although we introduce the Barndorff-Nielsen and Shephard model as an typical example of models with random coefficients, we do not discuss its LRM in this thesis. As a continuation of this thesis, we consider LRM for the Barndorff-Nielsen and Shephard model in Arai and Suzuki [6].

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