MEASURING MARGINAL UTILITY: THE PROBLEM OF IRVING FISHER REVISITED

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Abstract The method of measuring marginal utility devised by Irving Fisher is discussed. Deterministic nature of the method is illuminated and an attempt for statistical extension is made. The discussion is a step towards filling the gap between the classical methods of measuring marginal utility and the modern econometric methods of estimating utility function.

1. INTRODUCTION

1.1. The rise of the utility theory also gave rise to a discussion on necessity and possibility of empirical measurement of the notion of utility, Jevons [4]. Fisher [1] and Frisch ([2], [3]) developed Jevons' idea by elaborating practical methods for utility measurement. Since then, the notions of the indifference curve and the marginal rate of substitution have replaced the utility function and the marginal utility in the theory of consumers' behavior. It appears now that the work of Fisher and Frisch is only of a historical interest in the field of Econometrics.

In applications of Fisher and Frisch method, we are liable to get confused with inconsistent measurements provided by their method: Applying Fisher method in its original form to two Engel curves at different time points (or places) we can obtain a measurement of a marginal utility curve. If one more independent Engel curve is available, we end up with three measurements of the curve. These measured curves have to be identical in principle or close with each other at least approximately. Actual measurements, however, do not show this property of identity or close approximation. This inconsistency of measurements by Fisher and Frisch method may have been one of the reasons why one casts doubt upon the validity of their method.

1.2. The problem of utility measurement has gradually earned a modern outlook through the work of Wald [9], Stone [8], Parks [6] and others, which incorporate methods of statistical inference with econometric measurement of utility functions. Principles of modern methods of utility measurement or statistical estimation of utility function are no different from those of the classical methods in the sense that the equilibrium equation of the theory of consumers'

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behavior is a common basis of the classical methods and the modern econometric methods. Statistical principles and the equilibrium equation together constitute modern fashion of econometric estimation of utility functions.

Fisher and Frisch concentrated their discussion mainly on how to utilize the equilibrium equation for the measurement which should have been conducted within the limited availability of statistical data. Particularly for Fisher, a consideration of statistical technique was left for a future study. And Frisch's analysis of statistical method was not necessarily based on modern concepts developed after his time. It may be thought that Fisher and Frisch method is sensitive to statistical error and small error cause large variation of their measurement. And it is felt that a certain statistical principle should be incorporated to their method.

1.3. In this article, we clarify the basic principle on which Fisher based his practical method of utility measurement, and try to find out the reason why his method yields inconsistent results. Besides, a suggestion for a statistical extension is given so that his method is applicable to general situations.

2. FISHER'S METHOD

2.1. We consider a two-good model of the consumer's behavior. Let q_F and q_H be quantities consumed for goods F and H respectively, and p_F and p_H be their prices. Total expenditure E satisfies the budget equation,

$$p_F q_F + p_H q_H = E\,. \tag{2.1}$$

We set the functions

$$u_F = u_F(q_F)\,, \qquad u_H = u_H(q_H) \tag{2.2}$$

to represent marginal utilities of the goods F and H, where the functions u_F and u_H are assumed to be dependent only on q_F and q_H respectively.

The marginal utility of money λ is a function of p's and E

$$\lambda = \lambda(p_F, p_H, E)\,. \tag{2.3}$$

The first order condition for the utility maximization is

$$u_F/p_F = u_H/p_H\,. \tag{2.4}$$

We can rewrite this equation into the form

$$q_H = f(q_F : p_F, p_H)\,, \tag{2.5}$$

which corresponds to the expansion path given p_F and p_H fixed. By C we denote the expansion path (2.5) under the relative price $\phi = p_F/p_H$.

2.2. Fisher devised a procedure for measuring the marginal utility (of money) under the assumptions:

(a) A set of budget data, which represents two expansion paths C_1 and C_2 under two different relative price situations ϕ_1 and ϕ_2, is available.

(b) The utility function underlying the budget data is uniform.

(c) The utility function is additive so that the marginal utility functions take the form of (2.2).

The set of assumptions (a), (b), (c) is a sufficient condition for the possibility of utility measurement. Frisch presented different sufficient conditions.

2.3. An actual problem we encounter in measurement work is not whether the assumptions (a), (b), (c) are really sufficient condition, but whether the hypotheses (b), (c) are empirically valid to explain the variations in the data (a).

Morgan [5] takes up Fisher method, regarding the expansion path of Boston as C_1 and food as F. Substituting the expansion paths of several cities for C_2 and several consumption items for H, he gets number of combinations of data and therefore obtains different measurements of the marginal utility of money in Boston. The results which does not show much uniformity among the measurements may contribute to doubts on the validity of Fisher method or even on the possibility of utility measurement.

However, if we want to determined empirically an additive utility function which is intended to explain budget data concerning more than two goods and more than two places, Fisher method should be extended to be applicable to such a complicated case.

2.4. We set another assumption:

(d) The expansion path C is approximated by a linear equation,

$$q_H=\alpha+\beta q_F\,. \tag{2.6}$$

This assumption is not necessary for Fisher method but this kind of operationality is required in actual analyses.

The linearity assumption (d) in addition to the additivity assumption (c) implies that the utility function belongs to Pollak family of utility functions, Pollak [6].

2.5. In Fig. 1, let C_1 and C_2 be expansion paths of time 1 and 2. From the fixed initial point a we determine points b, c, $\cdots$, so that the following equations are satisfied,

$$\begin{aligned} &q_F(a)=q_F(b)\,, &\quad q_H(b)=q_H(c)\,,\\ &q_F(c)=q_F(d)\,, &\quad q_H(d)=q_H(e)\,,\\ &\quad\cdots\quad, &\quad \cdots\quad. \end{aligned} \tag{2.7}$$

where $q_F(x)$ and $q_H(x)$ are coordinates of q_F and q_H at point $x=a$, b, $\cdots$.

The first order condition for utility maximization at the point a is

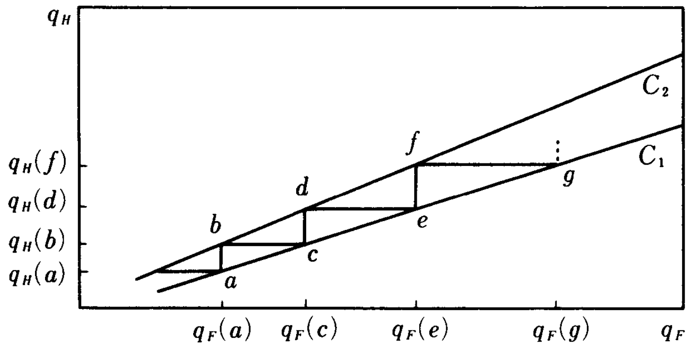

Fig. 1. Two expansion paths and equal consumption path.

$$u_F(q_F(a))/p_{F1}=u_H(q_H(a))/p_{H1}=\lambda_1(a)\,, \tag{2.8}$$

where p_{Ft}, p_{Ht} are prices at time t, and $\lambda_t(a)$ is marginal utility of money at equilibrium point a. Similarly, the equations hold at points b, c, $\cdots$,

$$\begin{aligned} u_F(q_F(b))/p_{F2}&=u_H(q_H(b))/p_{H2}=\lambda_2(b)\,,\\ u_F(q_F(c))/p_{F1}&=u_H(q_H(c))/p_{H1}=\lambda_1(c)\,,\\ &\cdots\cdots \end{aligned} \tag{2.9}$$

In view of the additivity and (2.7) we have

$$\begin{aligned} u_F(q_F(a))&=u_F(q_F(b))\,, & u_H(q_H(b))&=u_H(q_H(c))\,,\\ u_F(q_F(c))&=u_F(q_F(d))\,, & u_H(q_H(d))&=u_H(q_H(e))\,,\\ &\cdots\cdots \end{aligned} \tag{2.10}$$

Then we get relationships between the marginal utilities of money at a and b, from (2.9) and (2.10),

$$\lambda_1(a)p_{F1}=u_F(q_F(a))=u_F(q_F(b))=\lambda_2(b)p_{F2}\,, \tag{2.11}$$

and therefore

$$\lambda_2(b)=\lambda_1(a)p_{F1}/p_{F2}\,. \tag{2.12}$$

Similarly, we obtain the equations,

$$\begin{aligned} \lambda_1(c)&=\lambda_2(b)p_{H2}/p_{H1}\,,\\ \lambda_2(d)&=\lambda_1(c)p_{F1}/p_{F2}\,,\\ &\cdots\cdots \end{aligned} \tag{2.13}$$

Normalizing as $\lambda_1(a)=1$ and denoting p_{Ft}/p_{Ht} by ϕ_t, we can get a table for calculating marginal utility of money, λ_1 and λ_2 respectively along the expansion paths C_1 and C_2,

(2.14)

x	$\lambda_1(x)$	y	$\lambda_2(y)$
a	1		
		b	$1\ p_{F1}/p_{F2}$
c	(ϕ_1/ϕ_2)		
		d	$(\phi_1/\phi_2)(p_{F1}/p_{F2})$
e	$(\phi_1/\phi_2)^2$		
		f	$(\phi_1/\phi_2)(p_{F1}/p_{F2})^2$

2.6. Since the marginal utility equation (2.4) holds at each point, we can calculate u_F and u_H by the equations,

(2.15) $$u_F=\lambda p_F\,, \qquad u_H=\lambda p_H$$

where λ is given by (2.14).

Thus, given the expansion paths C_1, C_2 and the relative prices ϕ_1, ϕ_2, we first determine the points $a, b, c, \cdots$, then calculate the marginal utilities u_F, u_H at these points according to the following tables;

(2.16)

x	$u_F(x)$
$q_F(a)=q_F(b)$	$p_{F1}\,1$
$q_F(c)=q_F(d)$	$p_{F1}(\phi_1/\phi_2)$
$q_F(e)=q_F(f)$	$p_{F1}(\phi_1/\phi_2)^2$
$\cdots$	$\cdots$

(2.17)

y	$u_H(y)$
$q_H(a)$	$p_{H1}\,1$
$q_H(b)=q_H(c)$	$p_{H1}(\phi_1/\phi_2)$
$q_H(d)=q_H(e)$	$p_{H1}(\phi_1/\phi_2)^2$
$\cdots$	$\cdots$

2.7. The measurements, $p_{F1}(\phi_1/\phi_2)^i$, $p_{H1}(\phi_1/\phi_2)^i$ are functions of the exogenous prices and are therefore free from errors in the sense of 'shock,' if we assume that the measurement is carried out within a framework of a shock model. Estimation problems occur when we determine points $a, b, c, \cdots$, or essentially when we fit linear expansion paths C_1, C_2 to badget data. The fitted linear equations are subject to sampling errors, so are the determined points $a, b, c, \cdots$.

2.8. We write the fitted linear paths (regression equations by the method of least squares, for instance), for the budget data of time t, as

$$q_H = \alpha_t + \beta_t q_F, \qquad t = 1, 2. \tag{2.18}$$

Letting $q_F^0 = q_F(a)$, $q_F^1 = q_F(c)$, $q_F^2 = q_F(e)$, and so on, we have

$$\begin{aligned} q_H &= \alpha_1 + \beta_1 q_F^{i+1}, \\ q_H &= \alpha_2 + \beta_2 q_F^{i}, \qquad i = 0, 1, 2, \cdots, \end{aligned} \tag{2.19}$$

or

$$q_F^{i+1} = (\alpha_2 - \alpha_1)/\beta_1 + (\beta_2/\beta_1) q_F^i, \qquad i = 0, 1, 2, \cdots. \tag{2.20}$$

The solution of the difference equation is, if $\beta_1 \neq \beta_2$,

$$q_F^i = (\alpha_2 - \alpha_1)/(\beta_1 - \beta_2) + (q_F^0 - (\alpha_2 - \alpha_1)/(\beta_1 - \beta_2))(\beta_2/\beta_1)^i, \tag{2.21}$$

or if $\beta_1 = \beta_2$,

$$q_F^i = i(\alpha_2 - \alpha_1)/\beta_1 + q_F^0. \tag{2.22}$$

In a similar way, letting $q_H^0 = q_H(a)$, $q_H^1 = q_H(b)$, $q_H^2 = q_H(d)$, and so on, we have the solution, if $\beta_1 \neq \beta_2$,

$$q_H^i = (\beta_1\alpha_2 - \beta_2\alpha_1)/(\beta_1 - \beta_2) + (q_H^0 - (\beta_1\alpha_2 - \beta_2\alpha_1)/(\beta_1 - \beta_2))(\beta_2/\beta_1)^i, \tag{2.23}$$

or if $\beta_1 = \beta_2$,

$$q_H^i = i(\alpha_2 - \alpha_1) + q_H^0, \qquad i = 0, 1, 2, \cdots. \tag{2.24}$$

For q_F^i given by (2.21) or (2.22), the measurement of its marginal utility is

$$u_F(q_F^i) = p_{F1}(\phi_1/\phi_2)^i, \tag{2.25}$$

and for q_H^i given by (2.23) or (2.24), the utility measurement is

$$u_H(q_H^i) = p_{H1}(\phi_1/\phi_2)^i. \tag{2.26}$$

3. DETERMINISTIC MEASUREMENT

3.1. The data on which our analysis in this and the following sections is based is a set of cross-sections of the years 1965 through 1973, *Family Income and Expenditure Survey* by Bureau of Statistics, Office of the Prime Minister Japan.

The goods F and H are identified as Food and Housing according to the FIES classification. The p_F and p_H are the corresponding price indexes. The q_F and q_H are the quantity indexes derived from the nominal statistics and the price index.

3.2. First fitting regression equations (Engel curves of F and H) to the cross-sectional data of time t by the least squares method, then eliminating the variable of the total expenditure E, we get estimates of coefficients, α_t and β_t, of the expansion path C_t. The estimates and the relative prices for the years 1965 through 1973 are given in Table 1, columns (1) and (5). Later on we call C_t the least squares expansion path.

TABLE 1. OBSERVED AND PREDICTED EXPANSION PATH

	(1)	(2)	(3)	(4)	(5)
t	α_t		a_t		ϕ_t
1965	−1619	−1683	−1482	3209	416/486
1966	−1664	−1664	−1420	3887	432/511
1967	−6456	−1663	−1427	3806	435/535
1968	−2832	−1706	−1559	2291	482/555
1969	1153	−1735	−1657	940	511/578
1970	−1895	−1776	−1798	−1373	557/615
1971	−2582	−1800	−1880	−2922	591/644
1972	−2397	−1792	−1852	−2365	613/671
1973	−1843	−1843	−2030	−6225	693/738
t	β_t		b_t		
1965	.2738	.2941	.2931	.1823	
1966	.2857	.2857	.2825	.1588	
1967	.4121	.2868	.2838	.1616	
1968	.3697	.3043	.3058	.2140	
1969	.2617	.3172	.3223	.2607	
1970	.3361	.3356	.3460	.3407	
1971	.3948	.3461	.3597	.3942	
1972	.3780	.3425	.3549	.3749	
1973	.3652	.3652	.3848	.5084	

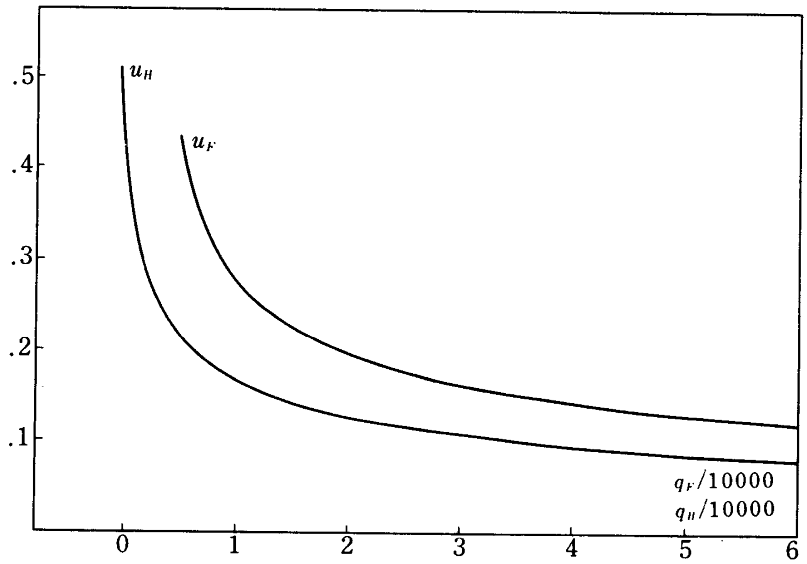

Fig. 2. Measured marginal utility curves for two goods.

Among several (9!/7! 2!) combinations of pair of the relative prices, the pair of ϕ_{1966} and ϕ_{1973} gives the largest difference. We therefore apply Fisher method of Section 2 to the pair of the least squares expansion paths of the years 1966 and 1973.

3.3. Using the method of Section 2, we obtain a measurement of marginal utilities of the goods Food and Housing for the two years, the result being illustrated in Fig. 2.

3.4. From the measured curves and the relative price data, we can predict expansion paths, $\hat{C}_t$, for the remaining seven years under different price situation. Comparatively well predicted expansion paths for the years 1965 and 1970 are given in Fig. 3a and Fig. 3b. It will be shown that for the years 1966 and 1973 the prediction $\hat{C}_t$ and observed C_t coincides and that is why we call the method deterministic.

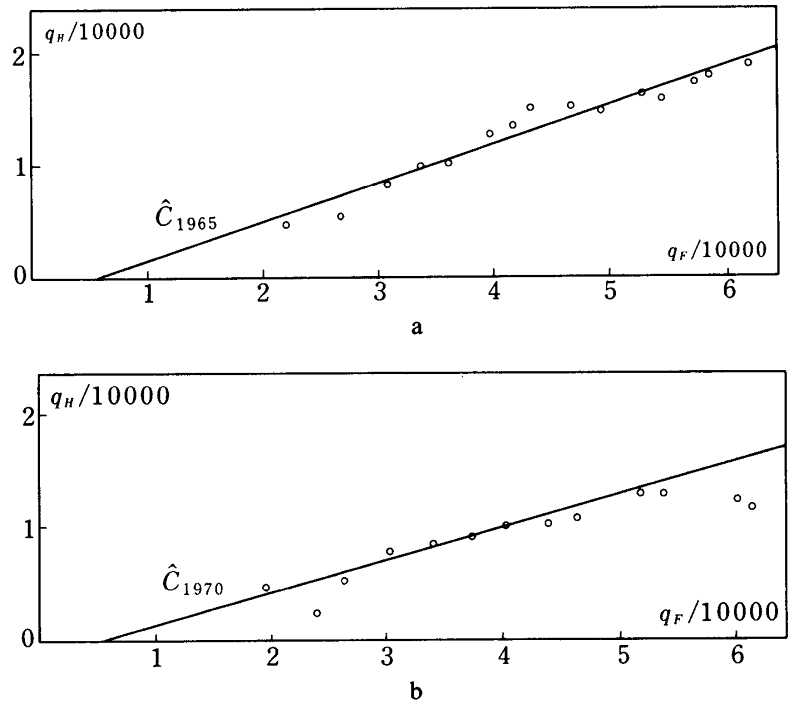

Fig. 3. Observed expansion paths and predicted expansion paths for 1965 and 1970.

3.5. It appears that the prediction $\hat{C}_{1965}$ based on the measurement from the data of 1966 and 1973 approximates the observed C_{1965} fairly closely. Therefore it is thought reasonable to measure the utility curve from the pair of 1965 and 1966 data. But the least squares expansion paths C_{1965} and C_{1966} and the relative prices ϕ_{1965} and ϕ_{1966} turn out to bear a relation like the one in Fig. 4. We can not find any regular utility function which yields the expansion paths C_1 and C_2 con-

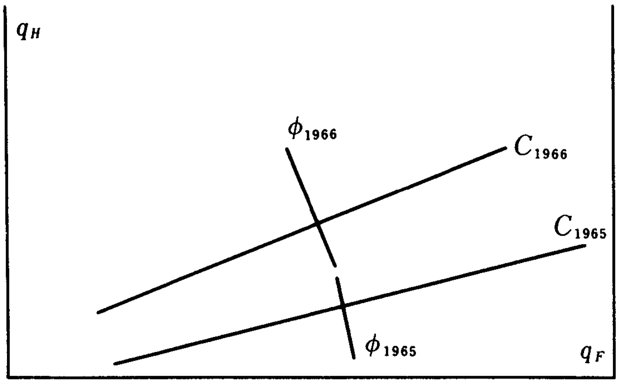

Fig. 4. Non-integrable expansion paths.

sistently with utility maximization under the price situations ϕ_1 and ϕ_2.

Among the nine years, we have some pairs of the expansion paths which cross at some point in the observation range. Fisher method dose not work for such cases.

Even leaving aside these pathological (non-integrability) cases, we can not see much uniformity among the measurements from various combinations of C's.

4. ANALYTICAL FITTING

4.1. If $\beta_1 \neq \beta_2$ and $q_F{}^0 > (\alpha_2 - \alpha_1)/(\beta_1 - \beta_2)$, the succesive $q_F{}^i$'s are given by (2.21) and corresponding values of utilities are given by (2.25). Eliminating the discrete variable i, we obtain, from (2.21) and (2.25),

(4.1) $$u_F = p_{F1}(q_F{}^0 - (\alpha_2 - \alpha_1)/(\beta_1 - \beta_2))^{-\varepsilon}(q_F - (\alpha_2 - \alpha_1)/(\beta_1 - \beta_2))^{\varepsilon},$$

where

(4.2) $$\varepsilon = \log(\phi_1/\phi_2)/\log(\beta_2/\beta_1).$$

Similarly, we obtain, from (2.23) and (2.26),

(4.3) $$u_H = p_{H1}(q_H{}^0 - (\beta_1\alpha_2 - \beta_2\alpha_1)/(\beta_1 - \beta_2))^{-\varepsilon}(q_H - (\beta_1\alpha_2 - \beta_2\alpha_1)/(\beta_1 - \beta_2))^{\varepsilon}$$

Dividing both sides of (4.1) and (4.3) by $p_{H1}\beta_1{}^{-\varepsilon}(q_F{}^0 - (\alpha_2 - \alpha_1)/(\beta_1 + \beta_2))^{-\varepsilon}$ and recalling that $q_H{}^0 = \alpha_1 + \beta_1 q_F{}^0$, we get

(4.4) $$u_F{}^* = \phi_1\beta_1{}^{\varepsilon}(q_F - (\alpha_2 - \alpha_1)/(\beta_1 - \beta_2))^{\varepsilon}, \quad u_H{}^* = (q_H - (\beta_1\alpha_2 - \beta_2\alpha_1)/(\beta_1 - \beta_2))^{\varepsilon},$$

which is called the normalized measurement.

4.2. The first order condition under the prices p_{Ft}, p_{Ht} is

(4.5) $$u_F{}^*/p_{Ft} = u_H{}^*/p_{Ht},$$

which reduces to the equation

(4.6) $$q_H = a_t + b_t q_F\,,$$

where

(4.7) $$\log b_t = (\log(\phi_1/\phi_t)/\log(\phi_1/\phi_2))\log\beta_2 - (\log(\phi_2/\phi_t)/\log(\phi_1/\phi_2))\log\beta_1\,,$$
$$a_t = ((\beta_1 - b_t)\alpha_2 - (b_t - \beta_2)\alpha_1)/(\beta_1 - \beta_2)\,.$$

Equation (4.6) is the predicted expansion path $\hat{C}_t$ under the price situation p_{Ft}, p_{Ht}, the prediction being based on the measurement from the least squares expansion paths C_1 and C_2. It is seen that the predicted coefficients, $\log b_t$ and a_t, are weighted averages of, respectively, $\log\beta$ and α.

The coefficients a_t and b_t of the prediction $\hat{C}_t$ are given in Table 1, Column (2).

4.3. From (4.6) and (4.7), we see that if $\phi_t = \phi_1$ then $(b_t, a_t) = (\beta_1, \alpha_1)$, and if $\phi_t = \phi_2$ then $(b_t, a_t) = (\beta_2, \alpha_2)$. Thus, in our linear system, the prediction $\hat{C}_t$ by Fisher method exactly coincides to the two least squares expansion paths C_1, C_2 used for the measurement of $u_F{}^*$, $u_H{}^*$.

In other words, *given the data* $D_1 \equiv (C_1, \phi_1)$ *and* $D_2 \equiv (C_2, \phi_2)$, *we can construct functions* $u_F{}^*$, $u_H{}^*$ *such that the equations* $u_F{}^*/p_{F1} = u_H{}^*/p_{H1}$ *and* $u_F{}^*/p_{F2} = u_H{}^*/p_{H2}$ *are the* C_1 *and* C_2. *Fisher method is an algorithm for constructing the* $u_F{}^*$ *and* $u_H{}^*$.

What will happen when the third data $D_3 \equiv (C_3, \phi_3)$ is available? We will have three measurements of $u_F{}^*$, $u_H{}^*$ according to the three pairs, (D_1, D_2), (D_1, D_3) and (D_2, D_3). If the data D_3 satisfies (4.6) and (4.7), then the three measurements must be identical.

4.4. For the case with condition that $\beta_1 \neq \beta_2$ and $q_F{}^0 < (\alpha_2 - \alpha_1)/(\beta_1 - \beta_2)$, we have the normalized measurement

(4.8) $$u_F{}^* = \phi_1\beta_1{}^\varepsilon((\alpha_2 - \alpha_1)/(\beta_1 - \beta_2) - q_F)^\varepsilon\,,$$
$$u_H{}^* = ((\beta_1\alpha_2 - \beta_2\alpha_1)/(\beta_1 - \beta_2) - q_H)^\varepsilon\,.$$

The prediction equation for this case is also given by (4.6) and (4.7).

4.5. For the case with $\beta_1 = \beta_2$, the normalized measurement is given by

(4.9) $$u_F{}^* = \phi_2^{\alpha_1/(\alpha_1 - \alpha_2)} \exp(\log(\phi_1/\phi_2)q_F/(\alpha_2 - \alpha_1))\,,$$
$$u_H{}^* = \phi_1^{\alpha_2/(\alpha_1 - \alpha_2)} \exp(\log(\phi_1/\phi_2)q_H/(\alpha_2 - \alpha_1))\,.$$

The prediction equation is given as

(4.10) $$q_H = a_t + b_t q_F\,,$$

where

(4.11) $$b_t = \beta_1 = \beta_2\,,$$
$$a_t = (\log(\phi_1/\phi_t)\alpha_2 - \log(\phi_2/\phi_t)\alpha_1)/\log(\phi_1/\phi_2)\,.$$

5. STATISTICAL MEASUREMENT

5.1. Under the linearity and additivity, the measurement of Fisher method reduces to the utility function (4.4), (4.8) or (4.9). We here reverse the preceding discussion by starting from a specification of marginal utility function.

We reparameterize the function (4.4) as

$$(5.1)\qquad \begin{aligned} u_F{}^* &= k_F(q_F - l_F)^v , \\ u_H{}^* &= k_H(q_H - l_H)^v , \end{aligned}$$

where the parameters k, l, v are to be estimated. The expansion path under relative price ϕ_t is

$$(5.2)\qquad q_H = \alpha_t + \beta_t q_F , \qquad t = 1, 2, \cdots, T ,$$

where

$$(5.3)\qquad \begin{aligned} \beta_t &= (k_F/\phi_t k_H)^{1/v} , \\ \alpha_t &= l_H - \beta_t l_F . \end{aligned}$$

5.2. Suppose that we have the data of ϕ_t and the estimated α_t, β_t from cross-sectional budget data at time t. The estimates α_t and β_t are subject to sampling error. From (5.3), we set statistical equations, for the T estimates of α_t and β_t,

$$(5.4)\qquad \begin{aligned} \log \beta_t &= \frac{1}{v} \log k + \frac{1}{v} \log (1/\phi_t) , \\ \alpha_t &= l_H + l_F(-\beta_t) , \qquad t = 1, 2, \cdots, T , \end{aligned}$$

where disturbance terms standing for the sampling errors of α_t, β_t are omitted, and $k = k_F/k_H$.

The least squares principle applied to (5.4) suggests a set of estimates of the parameters

$$(5.5)\qquad \begin{aligned} \theta_1 &= 1/v , \\ \theta_0 &= (\log k)/v , \end{aligned}$$

as

$$(5.6)\qquad \begin{aligned} \hat{\theta}_1 &= \frac{\sum_{t=1}^{T} (\log (1/\phi_t) - \overline{\log (1/\phi)})(\log \beta_t - \overline{\log \beta})}{\sum_{t=1}^{T} (\log (1/\phi_t) - \overline{\log (1/\phi)})^2} \\ \hat{\theta}_0 &= \overline{\log \beta} - \hat{\theta}_1 \, \overline{\log (1/\phi)} , \end{aligned}$$

where $\bar{x} = \sum x_t/T$.

Since equations (5.4) are simultaneous of a recursive type, we first calculate

$$\hat{\beta}_t = \exp(\hat{\theta}_0 + \hat{\theta}_1 \log(1/\phi_t)) , \tag{5.7}$$

then apply least squares method to the second set of equations of (5.4). This results in the estimates of l_F and l_H,

$$\begin{aligned} \hat{l}_F &= \sum_t (-\hat{\beta}_t + \bar{\hat{\beta}})(\alpha_t - \bar{\alpha}) \Big/ \sum_t (-\hat{\beta}_t + \bar{\hat{\beta}})^2 , \\ \hat{l}_H &= \bar{\alpha} + \hat{l}_F \bar{\hat{\beta}} \end{aligned} \tag{5.8}$$

Finally, from (5.5) we derive the estimates of v and k as

$$\begin{aligned} \hat{v} &= 1/\hat{\theta}_1 , \\ \hat{k} &= \exp(\hat{\theta}_0/\hat{\theta}_1) . \end{aligned} \tag{5.9}$$

If we have $T(>2)$ cross-sectional data, α_t and β_t, overall estimation of v, k, l_F and l_H is possible by using the T cross-section expansion paths in terms of α_t, β_t in spite of the deterministic measurement using only two expansion paths.

5.3. If $T=2$, then the estimates given above become deterministic rather than statistical. Since in this case identities like

$$\begin{aligned} \log(1/\phi_1) - \overline{\log(1/\phi)} &= (\log(1/\phi_1) - \log(1/\phi_2))/2 , \\ \log \beta_1 - \overline{\log \beta} &= (\log \beta_1 - \log \beta_2)/2 , \end{aligned} \tag{5.10}$$

hold, it follows that

$$\begin{aligned} \hat{v} &= (\log \phi_1 - \log \phi_2)/(\log \beta_2 - \log \beta_1) , \\ \hat{k} &= \phi_1 \beta_1^{\hat{v}} , \\ \hat{l}_F &= (\alpha_2 - \alpha_1)/(\beta_1 - \beta_2) , \\ \hat{l}_H &= (\beta_1 \alpha_2 - \beta_2 \alpha_1)/(\beta_1 - \beta_2) . \end{aligned} \tag{5.11}$$

Thus when $T=2$, our statistical measurement, $\hat{v}$, $\hat{k}$, $\hat{l}_F$, $\hat{l}_H$, is identical to Fisher measurement (4.2) and (4.4) under the assumed linear system.

5.4. The information about the nine least squares expansion paths and the relative prices of the years 1965 through 1973, transformed as $\log \beta_t$ and $\log(1/\phi_t)$, is shown in Fig. 5. The measurement of Section 3 is obtained by fitting the regression (5.4) deterministically to the two points ($\log \beta_{1966}$, $\log(1/\phi_{1966})$) and ($\log \beta_{1973}$, $\log(1/\phi_{1973})$), therefore the estimate $\hat{v}$ is negative. Whereas fitting the regression to points ($\log \beta_{1965}$, $\log(1/\phi_{1965})$) and ($\log \beta_{1966}$, $\log(1/\phi_{1966})$) gives contradicting positive estimate as easily seen in Fig. 5. It is also observed in Fig. 5 that the plot ($\log \beta_{1970}$, $\log(1/\phi_{1970})$) is close to the deterministically fitted line, and

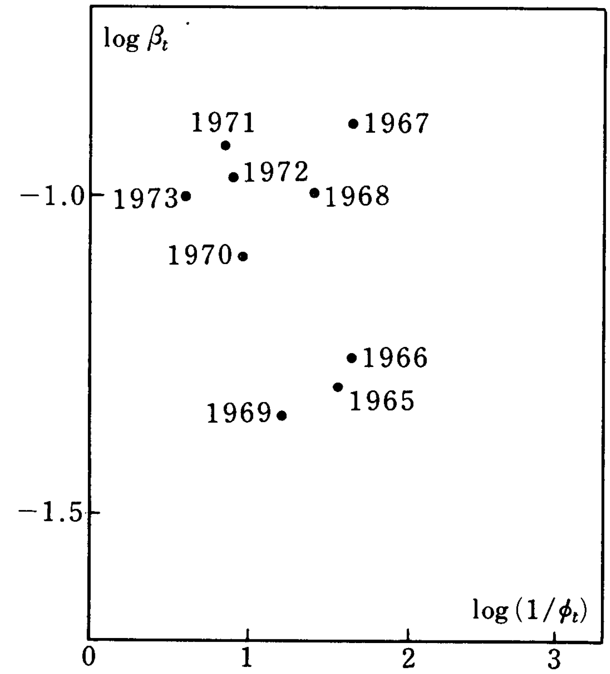

Fig. 5. Time series relation between the coefficient of expansion path and the relative price.

the prediction $\hat{C}_{1970}$ is close to observed C_{1970}.

5.5. The parallel discussion with the preceding one is possible if we start from a specification;

$$u_F{}^* = k_F(l_F - q_F)^v\,, \qquad u_H{}^* = k_H(l_H - q_H)^v\,, \tag{5.12}$$

which is a reparameterization of (4.8), or if we start from a specification;

$$u_F{}^* = k_F \exp l_F q_F\,, \qquad u_H{}^* = k_H \exp l_H q_H\,, \tag{5.13}$$

a reparameterization of (4.9).

5.6. The plot of $(\alpha_t, -\beta_t)$ shows that the $(\alpha_{1967}, -\beta_{1967})$ is exceptional, we therefore apply the statistical method to the remaining eight years. Resulting regression estimates by the method (5.6) and (5.8) are

$$\log \beta_t = -0.76997 - 2.94125 \log (1/\phi_t)\,, \quad r = -0.673 \qquad \alpha_t = 267.261 + 5970.210\,(-\hat{\beta}_t)\,, \quad r = 0.173\,. \tag{5.14}$$

And the reduced utility measurement is

$$(5.15)\qquad \begin{aligned} u_F^* &= 1.2992\ (q_F - 5970)^{-0.33999}\ , \\ u_H^* &= 1.0 \qquad (q_H - \ 267)^{-0.33999}\ . \end{aligned}$$

The prediction based on this utility function is given in Table 1, Column (3).

From the eight points in Fig. 5 the deterministic measurement is possible in 8!/6! 2! ways. It is seen, however, that the deterministic method might provide unstable values of $\hat{v}$ including negative and positive ones.

5.7. The correlation coefficients of the regression (5.14) are low, so that the uniformity hypothesis for the utility function during the eight years seems to be rejected. We move on to carry utility measurement by restricting ourselves to the four years from 1969 to 1972. We get regression estimates

$$(5.16)\qquad \begin{aligned} \log \beta_t &= 0.02044 - 11.07738 \log (1/\phi_t)\ , \qquad r = -.9987\ , \\ \alpha_t &= 8481.245 + 28927.166\ (-\hat{\beta}_t)\ , \qquad r = .9670\ , \end{aligned}$$

and the reduced utility measurement,

$$(5.17)\qquad \begin{aligned} u_F^* &= 0.99816\ (q_F - 28927.)^{-0.09027}\ , \\ u_H^* &= 1.0 \qquad (q_H - \ 8481.)^{-0.09027}\ . \end{aligned}$$

The prediction from this utility function is given in Table 1, Column (4). The result shows that the utility function differs from commonly employed Geary-Stone type utility function. The measured utility function yields the expansion paths fitting well to the data through 1969 to 1972. Whereas, the predicted expansion paths for the years before 1968 and after 1973 are regarded unsatisfactory. That is, an additive utility function is so restrictive that we can not explain the variations in the data during long time periods. Otherwise the assumption of constant utility function is to be rejected.

6. CONCLUDING REMARKS

Fisher's method can be termed as nonparametric method of utility measurement, since the method does not assume any functional form of utility function but additivity, and the measurement is carried within a wide class of maintained hypotheses. Modern parametric method for estimating utility function is constructed after assuming that the functional form of the utility function is known. The method given in this paper assumes additivity of utility function and linearity of Engel curve. These assumptions are more restrictive relative to those of Fisher's and less restrictive than those of modern parametric methods. Therefore, our method can empirically identify the form of the utility function, which belongs to Pollak family of utility functions, without assuming the functional form completely.

Confronted with many discrepant observations, we are accustomed to take the mean of the number of observations. This is a role of statistics: Fisher's method in its original form yields many discrepant utility measurements when applied to time series of cross section budget data. For such a case we should reduce many measurements into a unique measurement by taking their mean in the way suggested in our analysis or in some other way.

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