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The Nonlinear Bending of Circular Rings under Uniform External Pressure

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Masao MIZUNO*

Abstract

The post-buckling form of circular rings under uniform external pressure is analyzed and the closed-form solution is given in terms of elliptic integrals. In particular, the value of the critical pressure is obtained for the limiting case of the solution.

I. Introduction

The nonlinear bending of circular rings of uniform section pushed or pulled out by a pair of diametrically opposite forces in its plane was solved by Yokota 1 in 1924. Sontag 2, Frisch-Fay 3, and Pan 4 also published the solution to the same problem.

The problem of nonlinear bending of straight or circular-arc cantilevers under the normally and uniformly distributed load was solved by Mitchell 5 in 1959.

Because of nonlinearity, we must notice that the law of superposition cannot be applied to these problems.

In the present paper, the nonlinear bending of circular rings of uniform section under uniform external pressure is analyzed.

II. Non-linear Solution

Let $BAB'$ be half of a circular ring in post-buckling state, as shown in Fig. 1. It may be assumed that $BB'$ and $OA$ are axes of symmetry for the ring; then the action of the removed lower portion can be represented by a longitudinal compressive force $W$ and by a bending moment $M_0$ acting on each of the cross sections $B$ and $B'$. Let $w$ be the uniform normal pressure per unit length of the neutral line of the ring and $OA=a$, $OB=OB'=b$. Then the compressive force at $B$ and $B'$ is

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2) R. Sontag, Ingr.-Arch. 13 (1943) 380.
and the bending moment at any cross section \( P \) of the buckled ring is

\[
M = M_0 + W \frac{bh}{2} - \frac{w}{2} \frac{PB^2}{OB^{-2}} = M_0 + wbh - \frac{w}{2} c^2
\]

Now, considering the triangle \( PBO \),

\[
OP^2 = OB^2 + PB^2 - 2OB \cdot BH
\]

or

\[
wh - \frac{c^2}{2} = -\frac{1}{2} (r^2 - b^2).
\]

Substituting this in the expression (b) for the bending moment, we obtain

\[
M = M_0 - \frac{w}{2} (r^2 - b^2).
\]

Bernoulli-Euler's equation states that

\[
M = D \left( \frac{1}{R} - \frac{1}{\rho} \right),
\]

where \( D \) is the flexural rigidity of cross section of the ring, \( R \) the original radius of the ring, and \( \rho \) the radius of curvature of the neutral line at the arbitrary point \( P \). With the expression (c) for bending moment,

\[
\frac{1}{\rho} = \frac{1}{r} \frac{dv}{dr} \quad \text{and} \quad v = r \left[ 1 + \left( \frac{1}{r \frac{dr}{d\theta}} \right)^2 \right]^{-1/2},
\]

(2)
the differential equation for the deflection curve is
\[ \frac{dv}{dr} = \frac{w}{2D} (r^3 - b^2 r) - \frac{M_0}{2D r^2} - \frac{v}{R} \cdot \frac{r}{R}. \]

This can be integrated immediately to give
\[ v = b + \frac{w}{8D} \left( r^2 - b^2 \right) - \frac{M_0}{2D} \left( r^2 - b^2 \right) + \frac{1}{2R} \left( r^2 - b^2 \right). \] \( \text{Eqn. (e)} \)

The constant of integration is determined from the condition \( r = v = b \) at point B.

The unknown moment \( M_0 \) is given from the condition \( r = v = a \) at point A, i.e.
\[ \frac{M_0}{2D} = \frac{w}{8D} (a^2 - b^2) + \frac{1}{2R} - \frac{1}{a + b}, \]

and using the non-dimensional notations
\[ \frac{r}{L} = \gamma, \quad \frac{a}{L} = \alpha, \quad \frac{b}{L} = \beta \quad \text{and} \quad \kappa = \frac{wL^3}{2}, \quad \text{where} \quad L = \frac{\pi R}{2}, \]

\[ \frac{L}{2} \left( \frac{1}{R} - \frac{M_0}{2D} \right) = \frac{1}{\alpha + \beta} - \kappa (\alpha^2 - \beta^2). \] \( \text{Eqn. (f)} \)

Substituting this in the expression (e), we obtain
\[ \gamma \left[ 1 + \left( \frac{1}{r} \frac{d\gamma}{d\theta} \right)^2 \right]^{1/2} = \frac{L^2 + \alpha^2 \beta^2}{\alpha + \beta} + \kappa (\gamma^2 - \alpha^2) (\gamma^2 - \beta^2) \]

or
\[ d\theta = \frac{\gamma^2 + \alpha^2 \beta^2 + \kappa (\gamma^2 - \alpha^2) (\gamma^2 - \beta^2)}{2\kappa \gamma^2 \sqrt{R(\gamma^2)}} \, d\gamma, \] \( \text{Eqn. (g)} \)

where
\[ R(\gamma^2) = (\alpha^2 - \gamma^2) (\gamma^2 - \beta^2) \left[ (\gamma^2 - \beta^2)^2 \pm \gamma^2 \right], \quad (\alpha^2 > \gamma^2 > \beta^2) \]
\[ \beta = \frac{\alpha^2 + \beta^2}{2} - \frac{1}{\kappa (\alpha + \beta)}, \]

and
\[ q^2 = \pm (\alpha + \beta)^2 \left\{ \left( \frac{\alpha - \beta}{4} \right)^2 - \frac{1}{\kappa (\alpha + \beta)} \right\}; \quad \kappa (\alpha^2 - \beta^2) (\alpha - \beta) \leq 4. \] \( \text{Eqn. (h)} \)

Integrating the expression (g),
by putting
\[ I_s = \int_{\beta^2}^{2} \frac{d\gamma^2}{\sqrt{R(\gamma^2)}}, \quad I_1 = \int_{\beta^2}^{2} \frac{r^2 d\gamma^2}{\sqrt{R(\gamma^2)}}, \quad \text{and} \quad I_{-1} = \int_{\beta^2}^{2} \frac{r^2 d\gamma^2}{\gamma^2 \sqrt{R(\gamma^2)}}, \]

we get the equation of the curve \( BA \) in polar coordinates \( r, \theta \), i.e.
\[ \theta = \frac{1}{2} \left[ I_1 + \left\{ \frac{1}{\kappa (\alpha + \beta)} - (\alpha^2 + \beta^2) \right\} I_0 + \left\{ \frac{\alpha^2 \beta^2}{\kappa (\alpha + \beta)} + \alpha^2 \beta^2 \right\} I_{-1} \right] \]

In the equation (2), there are two unknowns \( \alpha \) and \( \beta \), and these are determined as follows:
The arc length \( s = \widehat{BP} \) is
\[
s = \int r \sqrt{1 + \left( \frac{1}{r} \frac{dr}{d\theta} \right)^2} \, d\theta
\]
or
\[
\frac{s}{L} = \int r \left[ 1 + \left( \frac{1}{r} \frac{dr}{d\theta} \right)^2 \right]^{1/2} \, d\theta = \int_{\beta^2}^{\gamma^2} \frac{dr^2}{2\kappa \sqrt{R(\gamma^2)}} ,
\]
and the condition of inextensibility is
\[
\kappa = \frac{1}{2} \int_{\beta^2}^{\gamma^2} \frac{dy^2}{\sqrt{R(\gamma^2)}} = \frac{1}{2} J_1 .
\] (3)

And, from (2)
\[
\frac{\pi}{2} = \frac{1}{2} \left[ J_1 + \left\{ \frac{1}{\kappa (\alpha + \beta)} - (\alpha^2 + \beta^2) \right\} J_0 + \left\{ \frac{2\beta}{\kappa (\alpha + \beta)} + \alpha^2 \beta^2 \right\} J_{-1} \right]
\]
or
\[
\pi - \left\{ \frac{1}{\kappa (\alpha + \beta)} - (\alpha^2 + \beta^2) \right\} 2\kappa = J_1 + \left\{ \frac{\alpha \beta}{\kappa (\alpha + \beta)} + \alpha^2 \beta^2 \right\} J_{-1} ,
\] (4)
where
\[
J_0 = \int_{\beta^2}^{\gamma^2} \frac{dy^2}{\sqrt{R(\gamma^2)}} , \quad J_1 = \int_{\beta^2}^{\gamma^2} \frac{x^2 \, dy^2}{\sqrt{R(\gamma^2)}} \quad \text{and} \quad J_{-1} = \int_{\beta^2}^{\gamma^2} \frac{dy^2}{\sqrt{R(\gamma^2)}} .
\] (i)

Let us consider now the condition that there will appear an inflection point at some portion of the quarter ring. From the expression (f), this condition is
\[
\kappa (\alpha^2 - \beta^2) (\alpha + \beta) > 1 .
\] (k)

From the expressions (h) and (k), we have to distinguish the following three cases:

Case 1; \( 0 < \kappa (\alpha^2 - \beta^2) (\alpha + \beta) \leq 1 ; \)
Case 2; \( \kappa (\alpha^2 - \beta^2) (\alpha + \beta) > 1 \) and \( \kappa (\alpha^2 - \beta^2) (\alpha - \beta) < 4 , \)
Case 3; \( \kappa (\alpha^2 - \beta^2) (\alpha - \beta) \geq 4 , \)
because of \( \kappa (\alpha^2 - \beta^2) (\alpha + \beta) > \kappa (\alpha^2 - \beta^2) (\alpha - \beta) . \)

II. Case 1; Non-inflexional Bending:
\[
0 > \kappa (\alpha^2 - \beta^2) (\alpha + \beta) \leq 1 ; \quad \text{[of course} \quad \kappa (\alpha^2 - \beta^2) (\alpha - \beta) < 4 \]}

As an aid in evaluating the integrals which occur in equations (2), (3) and (4) reference may be made to the tables by Byrd and Friedman.\(^6\)

The Nonlinear Bending of circular Rings under Uniform External Pressure

\[ I_o = \frac{1}{\sqrt{AB}} F(\varphi, k), \quad I_e = \frac{2}{\sqrt{AB}} K(k), \]

\[ I_1 = \frac{1}{\sqrt{AB}} \left[ \frac{\beta^2 a^2 - \alpha^2 b^2}{A - B} F(\varphi, k) + \frac{(\alpha^2 - \beta^2)}{2(A - B)} \left\{ \Pi \left( \varphi, \frac{(A - B)^2}{4AB} \right), k \right\} \right. \]
\[ - \left. 2 \sqrt{\frac{AB}{k^2(A + B) + k^1(A - B)^2}} \tan^{-1} \left( \frac{\sqrt{k^2(A + B)^2 + k^2(A - B)^2 \sin \varphi}}{2\sqrt{AB} \sqrt{1 - k^2 \sin^2 \varphi}} \right) \right] \]

\[ J_1 = \frac{2(\beta^2 a^2 - \alpha^2 b^2)}{\sqrt{AB} (A - B)} K(k) + \frac{(\alpha^2 - \beta^2) (A + B)}{2\sqrt{AB} (A - B)} \Pi \left( \varphi, \frac{(A - B)^2}{-4AB}, k \right), \]

\[ I_{-1} = \frac{A - B}{\alpha^2 A + \beta^2 B} F(\varphi, k) + \frac{(\beta^2 - \alpha^2)}{2a^2 b^2} \frac{(\beta^2 A + \alpha^2 B)}{(\beta^2 A - \alpha^2 B)} \times \left\{ \Pi \left( \varphi, \frac{(\beta^2 A - \alpha^2 B)^2}{-4a^2 \beta^2 AB}, k \right) \right\} \]
\[ - \frac{\beta^2 A - \alpha^2 B}{2a^2 b^2 \sqrt{AB} k^2(\beta^2 A + \alpha^2 B) + k^2(\beta^2 A - \alpha^2 B)} \]
\[ \tan^{-1} \left( \frac{\sqrt{k^2(\beta^2 A + \alpha^2 B) + k^2(\beta^2 A - \alpha^2 B) \sin \varphi}}{2a^2 b^2 \sqrt{AB} \sqrt{1 - k^2 \sin^2 \varphi}} \right) \]

and

\[ J_{-1} = \frac{2(A - B)}{\beta^2 A + \alpha^2 B} K + \frac{(\beta^2 - \alpha^2)}{2a^2 b^2} \frac{(\beta^2 A + \alpha^2 B)}{(\beta^2 A - \alpha^2 B)} \Pi \left( \varphi, \frac{(\beta^2 A - \alpha^2 B)^2}{-4a^2 \beta^2 AB}, k \right). \] (5)

In these equations the following notation has been used.

\[ \begin{align*}
A^2 &= (\alpha^2 - \beta^2) + q^2, \\
B^2 &= (\beta^2 - \alpha^2)^2 + q^2 \\
k^2 &= \frac{(\alpha^2 - \beta^2)^2 - (A - B)^2}{4AB}, \\
k_1^2 &= 1 - k^2, \\
(A \neq B, k^2 > \frac{(A - B)^2}{-4AB}, k^1 > \frac{(\beta^2 A - \alpha^2 B)^2}{-4a^2 \beta^2 AB}) \\
\cos \varphi &= \frac{(\alpha^2 - \beta^2) B - (\gamma^2 - \beta^2) A}{(\alpha^2 - \beta^2) B + (\gamma^2 - \beta^2) A}, \quad (0 \leq \varphi \leq \pi)
\end{align*} \]

(1)

For the limit of \( a \to \gamma, \beta \to \gamma, w \to w_{cr}, \) and \( k \to k_{cr}, \)

\[ \begin{align*}
A^2 &= B^2 = (\gamma^2 - p)^2 + q^2 = \frac{1 + 8 \kappa_{cr} \gamma^3}{4 \kappa_{cr}^3 \gamma^3}, \\
k &= 0.
\end{align*} \]

Substituting the equation (5) in the equation (3),

\[ \kappa_{cr} = \frac{1}{2} f_o = \frac{1}{\sqrt{AB}} K(\varphi) = -\frac{2 \kappa_{cr} \gamma}{\sqrt{1 + 8 \kappa_{cr} \gamma^3}} \times \frac{\pi}{2}. \]

Putting

\[ \gamma = \frac{r}{L} = \frac{R}{L} = \frac{2}{\pi}; \quad \sqrt{1 + 8 \kappa_{cr} \left( \frac{R}{L} \right)^3} = \gamma \pi = \frac{2}{\pi} \times \pi = 2 \]

and

\[ \kappa_{cr} = \frac{w_{cr} L^3}{8D}; \quad 1 + \frac{w_{cr} R^3}{D} = 4 \]

(5)
or
\[ \omega_{cr} = \frac{3D}{R^3}. \]

This result coincides with the critical pressure obtained from the theory of finite deformation neglecting squares of the small radial displacements \(^7\).

II. 2. Case 2; Inflexional Bending,
\[ \kappa (\alpha^2 - \beta^2) (\alpha - \beta) > 1 \quad \text{and} \quad \kappa (\alpha^2 - \beta^2) (\alpha - \beta) < 4. \]

We have also the equation (5) to the integrals which occur in equations (2), (3) and (4).

II. 3. Case 3; Inflexional Bending,
\[ \kappa (\alpha^2 - \beta^2) (\alpha - \beta) \geq 4. \]

In this case, we obtain the following expressions for (i) and (j).

\[ I_5 = \frac{2}{\sqrt{\alpha^2 - p - q}} \frac{F(\varphi, k)}{\sqrt{\beta^2 + p + q}} K(k), \quad J_5 = \frac{2}{\sqrt{\alpha^2 - p - q}} \frac{F(\varphi, k)}{\sqrt{\beta^2 - p + q}} K(k), \]

\[ I_4 = \frac{2}{\sqrt{\alpha^2 - p - q}} \frac{\beta^2 - p + q}{\sqrt{\beta^2 + p + q}} \left[ (p + q) F(\varphi, k) + (\beta^2 - p - q) II \left( \varphi, \frac{\alpha^2 - \beta^2}{\alpha^2 - p - q}, k \right) \right], \]

\[ J_4 = \frac{2}{\sqrt{\alpha^2 - p - q}} \frac{\beta^2 - p + q}{\sqrt{\beta^2 + p + q}} \left[ (p + q) K(k) + (\beta^2 - p + q) II \left( \frac{\alpha^2 - \beta^2}{\alpha^2 - p - q}, k \right) \right], \]

\[ I_{-5} = \frac{2}{\sqrt{\alpha^2 - p - q}} \frac{\beta^2 - p + q}{\sqrt{\beta^2 + p + q}} \left[ \frac{1}{p + q} F(\varphi, k) + \frac{1}{\beta^2 - p + q} \right] II \left( \frac{\alpha^2 - \beta^2}{\alpha^2 - p - q}, k \right), \]

and
\[ J_{-5} = \frac{2}{\sqrt{\alpha^2 - p - q}} \frac{\beta^2 - p + q}{\sqrt{\beta^2 + p + q}} \left[ \frac{1}{p + q} K(k) + \frac{1}{\beta^2 - p + q} \right] II \left( \frac{\alpha^2 - \beta^2}{\alpha^2 - p - q}, k \right). \]

(6)

In these equations the following notation has been used.

\[ k^2 = \frac{(\alpha^2 - \beta^2) \times 2q}{(\alpha^2 - p - q) (\beta^2 + p + q)} < 1 \]

and

\[ \sin \varphi = \sqrt{\frac{(\alpha^2 - p - q) (\gamma^2 - \beta^2)}{(\alpha^2 - \beta^2) (\gamma^2 - p - q)}}, \quad \left( 0 \leq \varphi \leq \frac{\pi}{2} \right). \]

III. Conclusion

The post-buckling form of circular rings under uniform external pressure is analyzed and the closed-form solution is given.

It is interesting that the solution is successively expressed in terms of elliptic integrals as in the case of circular rings under a pair of diametrically opposite forces.

In particular, the value of the critical pressure is obtained for the limiting case of the solution which coincides with the result from the elemental theory of elastic stability.