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New Proofs of Some Basic Theorems on Stationary Point Processes

Nariyuki Minami*

Summary—We give new proofs of three basic theorems on stationary point processes on the real line—theorems of Khintchine, Korolyuk, and Dobrushin. Moreover we give a direct construction of the Palm measure for a class of point processes which includes stationary ones as special cases.

1. Introduction

The purpose of this note is to give new proofs, based on a same simple idea, to some basic theorems on stationary point processes on the real line $\mathbb{R}$, as stated in standard treatises on point processes such as Daley and Vere-Jones (see §3.3 of [3]).

To begin with, let us introduce necessary definitions and notation. By $M_p$, we denote the set of all integer-valued Radon measures on $\mathbb{R}$. Namely $M_p$ is the totality of all measures $N(dx)$ on $\mathbb{R}$ such that for any bounded Borel set $B$, $N(B)$ is a non-negative integer. Let us call any such measure a counting measure. For a counting measure $N \in M_p$, let us define

$$X(t) := N((0, t]) \quad (t \geq 0), \quad : = -N((t, 0)) \quad (t < 0).$$

(1)

Then the function $X(t)$ is right-continuous, integer-valued, locally bounded and non-decreasing. Hence $X(t)$ is piecewise constant on $\mathbb{R}$ and the set $\Delta$, finite or countably infinite, of its points of discontinuity has no accumulation points other than $\pm \infty$. Thus the points in $\Delta$ can be ordered as

$$\cdots < x_{-1} < x_0 \leq 0 < x_1 < x_2 < \cdots,$$

so that if we let $m_n := X(x_n) - X(x_n - 0)$, then $N(dx)$ can be represented as

$$N(dx) = \sum_n m_n \delta_{x_n}(dx),$$

(2)

where $\delta_a$ denotes the unit mass placed at $a$. Each $m_n$ is a positive integer and is called

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the multiplicity of the point $x_n$. In general, either $N([0, \infty))$ or $N((\infty, 0))$ can be finite, in which case either $\{x_n\}_{n=0}^\infty$ or $\{x_n\}_{n=0}^\infty$ is a finite sequence. If in the former [resp. latter] case $\{x_n\}_{n=0}^\infty$ [resp. $\{x_n\}_{n=0}^\infty$] terminates with $x_\nu$, then we will set $x_n = \infty$ [resp. $x_n = -\infty$] for $n > \nu$ [resp. $n < \nu$]. When $m_n = 1$ for all $n$ such that $x_n \not= \pm \infty$, the counting measure $N$ is said to be simple. For each $N \in M_p$ with representation (2), let us associate a simple counting measure $N^*$ defined by

$$N^*(dx) = \sum_n \delta_{x_n}(dx).$$

In order to make $M_p$ a measurable space, we define $\mathcal{M}_p$ to be the $\sigma$-algebra of subsets of $M_p$ generated by all mappings of the form

$$M_p \ni N \mapsto N(B) \in [0, \infty]$$

for all Borel sets $B \subset \mathbb{R}$. Then we see that $x_n$, $m_n$ and $N^*$ are all measurable functions of $N$, as the following lemma shows.

**Lemma 1**  
(i) The set 

$$C := \{N \in M_p : N((\infty, 0]) = N((0, \infty)) = \infty\} = \{N \in M_p : x_n \text{ is finite for all } n\}$$

belongs to $\mathcal{M}_p$.

(ii) For each integer $n$, $x_n$ and $m_n$ are $\mathcal{M}_p$-measurable functions of $N$.

(iii) The mapping $M_p \ni N \mapsto N^* \in M_p$ is $\mathcal{M}_p/\mathcal{M}_p$-measurable.

**Proof.** (i) The assertion is obvious from the definition of $\mathcal{M}_p$, since we can write

$$C = \bigcap_{k=1}^\infty \bigcup_{n=1}^\infty \{N \in M_p : N((-n, 0]) > k, N((0, n]) > k\}.$$

(ii) The measurability of $x_1$ follows from the relation

$$\{N \in M_p : x_1 > t\} = \{N \in M_p : N(0, t] = 0\},$$

which holds for all $t \geq 0$. Now for each $k \geq 1$, define

$$x_1^{(k)} := \sum_{j=1}^\infty \mathbb{I}_{(j-1)/2^k, j/2^k}(x_1) + \mathbb{I}_{\{x_1 = \infty\}}.$$

Then we see that $x_1^{(k)}$ is measurable in $N$ and that $x_1^{(k)} \downarrow x_1$ as $k \to \infty$. By the right-continuity of $X(t) = N((0, t])$ at $t > 0$, we have, as $k \to \infty$,

$$\mathbb{I}_{\{x_1 < \infty\}} \cdot X(x_1^{(k)}) = \sum_{j=1}^\infty \mathbb{I}_{(j-1)/2^k, j/2^k}(x_1) X\left(\frac{j}{2^k}\right) \to X(x_1) = m_1,$$

which shows the measurability of $m_1$ in $N$.

Next let $\bar{X}(t) := X(t) - X(t \wedge x_1)$. This is measurable in $N$ for all $t \geq 0$, since
If we apply the above argument to $X(t)$ instead of $X(t)$, we can verify the measurability of $x_2$ and $m_2$ in $N$, and the argument can be iterated to give the measurability of all $x_n$ and $m_n$.

(iii) For each $j = 0, 1, 2, \ldots$ and $t > 0$, the sets

$$\{N \in M_p : N^\ast((0, t]) = j\} = \{N \in M_p : x_j \leq t < x_{j+1}\}$$

and

$$\{N \in M_p : N^\ast((-t, 0]) = j\} = \{N \in M_p : x_{-j} \leq t < x_{-j-1}\}$$

belong to $M_p$. Now for each $n \geq 1$, let $G_n$ be the class of all Borel subsets $B$ of $[-n, n]$ such that the mapping

$$M_p \ni N \mapsto N^\ast(B) \in [0, \infty)$$

is measurable. Then $G_n$ is seen to be a $\lambda$-system which contains the class of intervals

$$I := [(0, t] : 0 < t \leq t \cup \{(-t, 0] : 0 < t \leq n\}$$

which forms a $\pi$-system. Hence by Dynkin’s $\pi$-$\lambda$ theorem (see e.g. Durrett [2]), $G_n$ contains all Borel subsets of $[-n, n]$. Since $n \geq 1$ is arbitrary, and since we can write $N^\ast(B) = \lim_{n \to \infty} N^\ast(B \cap [-n, n])$, the mapping (5) is measurable for all Borel subsets of $\mathbb{R}$.

Remark 1. By an argument similar to (iii), it is easy to show that $M_p$ is generated by mappings $M_p \ni N \mapsto X(t)$ for all $t$, where $X(t)$ is defined in (1).

Definition 1 A point process $N_\omega$ is a random variable defined on a probability space $(\Omega, \mathcal{F}, P)$ and taking values in the measurable space $(M_p, \mathcal{M}_p)$.

Definition 2 A point process $N_\omega$ is said to be crudely stationary if for any bounded interval $I$ and for any $x \in \mathbb{R}$, $N_\omega(I)$ and $N_\omega(I + x)$ are identically distributed. Its mean density is the expectation value $m := E[N_\omega((0, 1])] \leq \infty$.

Definition 3 A point process $N_\omega$ is said to be stationary if for any $C \in \mathcal{M}_p$ and $x \in \mathbb{R}$, one has the identity

$$P(N_\omega(\cdot) \in C) = P(N_\omega(x + \cdot) \in C).$$
Obviously, $N_ω$ is crudely stationary if it is stationary.

Remark 2. By another application of $π$-$λ$ theorem, one can show without difficulty that $N_ω$ is stationary if and only if for any finite family of Borel subsets $B_1, \ldots, B_n$ of $R$, and of non-negative integers $k_1, \ldots, k_n$, the identity

$$P(N_ω(B_i) = k_i, i = 1, \ldots, n) = P(N_ω(x + B_i) = k_i, i = 1, \ldots, n)$$

holds for any $x ∈ R$.

2. Basic theorems and their proofs

Our argument is based on the following lemma, which is an immediate consequence of Definition 2.

Lemma 2 Let the point process $N_ω$ be crudely stationary. Then for any bounded interval $I$ and for any non-negative integer $k$,

$$P(N_ω(I) = k) = \int_0^1 P(N_ω(x + I) = k) dx = E \left[ \int_0^1 \mathbf{1}_{[N_ω((x, x + I))]} dx \right].$$

Proposition 1 (Khintchine’s theorem) For any crudely stationary point process $N_ω$, the limit

$$λ := \lim_{h \downarrow 0} \frac{1}{h} P(N_ω((0, h]) > 0)$$

exists and satisfies $λ \leq m$. $λ$ is called the intensity of the point process $N_ω$.

Proof. Let $N_ω$ be represented as (2) and define the point process $N_ω^*$ by (3). If we set $ν(ω) := N_ω^*(0, 1]$, it satisfies $x_{ν(ω)}(ω) ≤ 1 < x_{ν(ω)+1}(ω)$. Obviously we have

$$\{ x ∈ (0, 1) : N_ω([x, x + h]) > 0 \} = (0, 1] \cap \bigcup_{j=1}^{ν(ω)+1} [x_j(ω) - h, x_j(ω)) \bigcup_{j=1}^{ν(ω)+1} \left[ \left\{ J_j^ω(h) \setminus J_{j-1}^ω(h) \right\} \right].$$

where we have set $J_j^ω(h) := [x_j(ω) - h, x_j(ω))$ and $J_0 = 0$. Hence

$$\frac{1}{h} \int_0^1 \mathbf{1}_{[N_ω([x, x + h]) > 0]} dx = \frac{1}{h} \sum_{j=1}^{ν(ω)+1} \left[ (0, 1] \cap (J_j^ω(h) \setminus J_{j-1}^ω(h)) \right]$$

$$= \sum_{j=1}^{ν(ω)+1} \frac{1}{h} \left\{ \{1 ∧ x_j(ω)) \} - (0 ∨ x_{j-1}(ω) ∨ (x_j(ω) - h) \} \right\} \frac{1}{h} \left\{ \{1 ∧ x_j(ω)) \} - (0 ∨ x_{j-1}(ω) ∨ (x_j(ω) - h) \} \right\} .$$
where for a Borel subset $B$ of $\mathbb{R}$, $|B|$ denotes its Lebesgue measure and for a real number $a$, $a^+ := a \vee 0 = \max\{a, 0\}$ denotes its positive part. Now it is easy to see that for $1 \leq j \leq \upsilon(\omega)$,

$$\frac{1}{h} \{ 1 \wedge x_j(\omega) - \{ 0 \vee x_{j-1}(\omega) \vee (x_j(\omega) - h) \} \} \to 1$$

as $h \searrow 0$, and that for $j = \upsilon(\omega) + 1$,

$$\frac{1}{h} \{ 1 \wedge x_{\upsilon(\omega)+1}(\omega) - \{ 0 \vee x_{\upsilon(\omega)}(\omega) \vee (x_{\upsilon(\omega)+1}(\omega) - h) \} \},$$

is bounded by 1 and tends to 0 as $h \searrow 0$. Thus we can apply the monotone convergence theorem, the dominated convergence theorem and Lemma 2, to obtain

$$\frac{1}{h} \mathbb{P}(N_\omega((0, h]) > 0) = \mathbb{E} \left[ \frac{1}{h} \sum_{j=1}^{\upsilon(\omega)+1} \{ [0, 1] \cap (J^\omega_j \setminus J^\omega_{j-1}(h)) \} \right]$$

$$\to \mathbb{E} \left[ \sum_{j=1}^{\upsilon(\omega)} 1 \right] = \mathbb{E}[N^*_\omega((0, 1])],$$

as $h \searrow 0$. Thus the desired limit $\lambda$ exists and is equal to $\mathbb{E}[N^*_\omega((0, 1])]$. Clearly it satisfies the inequality $\lambda \leq \mathbb{E}[N_\omega((0, 1])] = m$.

**Corollary 1** If $N_\omega$ is simple, then $\lambda = m$. When $m < \infty$, the converse is also true.

**Proof.** $N_\omega$ is simple if and only if $N^*_\omega = N_\omega$ almost surely, which obviously implies $\lambda = m$. On the other hand, if $\lambda = m < \infty$, then

$$\mathbb{E}[N_\omega((0, 1]) - N^*_\omega((0, 1])] = m - \lambda = 0.$$

But $N_\omega((0, 1]) - N^*_\omega((0, 1]) \geq 0$ in general, so that $N_\omega((0, 1]) = N^*_\omega((0, 1])$ almost surely. The same argument is valid if the interval $(0, 1]$ is replaced by $(n, n + 1]$, so that $N_\omega((n, n + 1]) = N^*_\omega((n, n + 1])$ almost surely for all integers $n$, and the simplicity of $N_\omega$ follows.

**Remark 3.** In the treatise by Daley and Vere-Jones [3], for example, Proposition 1 is proved in the following way: If we define $\varphi(h) := \mathbb{P}(N_\omega((0, h]) > 0)$, then by the crude stationarity, we have for any positive $h_1$ and $h_2$,

$$\varphi(h_1 + h_2) = \mathbb{P}(N_\omega((0, h_1 + h_2]) > 0) = \mathbb{P}(N_\omega((0, h_1]) + N_\omega((h_1, h_1 + h_2]) > 0)$$

$$\leq \mathbb{P}(N_\omega((0, h_1]) > 0) + \mathbb{P}(N_\omega((h_1, h_1 + h_2]) > 0) = \varphi(h_1) + \varphi(h_2),$$

so that $\varphi(h)$ is a sub-additive function defined on $[0, \infty)$ satisfying $\varphi(0) = 0$. To show the existence of the intensity $\lambda$, it suffices to apply the following well known lemma.
Lemma 3  Let \( g(x) \) be a sub-additive function defined on \([0, \infty)\) such that \( g(0) = 0 \). Then one has
\[
\lim_{x \to 0^+} \frac{g(x)}{x} = \sup_{x > 0} \frac{g(x)}{x} \leq \infty.
\]
However, this argument does not provide the representation \( \lambda = \mathbb{E}[N^*_\omega((0, 1])] \), so that the proof of Corollary 1 requires some extra work. Our proof above is closer to that of Leadbetter [5]. See also Chung [1].

Definition 4  A crudely stationary point process \( N_\omega \) is said to be orderly when
\[
\mathbb{P}(N_\omega((0, h]) \geq 2) = o(h) \quad (h \searrow 0).
\]

Proposition 2 (Dobrushin’s theorem)  If a crudely stationary point process \( N_\omega \) is simple and if \( \lambda < \infty \), then \( N_\omega \) is orderly.

Proof. By Lemma 2, we can write
\[
\mathbb{P}(N_\omega((0, h]) \geq 2) = \mathbb{E}\left[ \int_0^h 1_{\{N_\omega((x, x+h]) \geq 2\}} \, dx \right].
\]
As can be seen from the proof of Proposition 1, we have
\[
\frac{1}{h} \int_0^h 1_{\{N_\omega((x, x+h]) \geq 2\}} \, dx \leq \frac{1}{h} \int_0^1 1_{\{N_\omega((x, x+h]) \geq 2\}} \, dx = \sum_{j=1}^{\nu(\omega)} 1_{\{(0, 1] \cap (J_{j\omega}^\epsilon(h) \setminus J_{j\omega}^\epsilon^{-1}(h)) \} + \frac{1}{h} \{0, 1] \cap (J_{\nu(\omega)+1}\omega(h) \setminus J_{\nu(\omega)}\omega(h))} \\
\leq N^*_\omega((0, 1]) + 1,
\]
and
\[
\lim_{h \searrow 0} \frac{1}{h} \int_0^h 1_{\{N_\omega((x, x+h]) \geq 2\}} \, dx = \mathbb{E}[\{j \in (0, 1] \mid \nu(\omega) \geq 2\}].
\]
Since \( \mathbb{E}[N^*_\omega((0, 1])] = \lambda < \infty \), we can apply the dominated convergence theorem, to obtain
\[
\lim_{h \searrow 0} \frac{1}{h} \mathbb{P}(N_\omega((0, h]) \geq 2) = \mathbb{E}[\{j \in (0, 1] \mid \nu(\omega) \geq 2\}],
\]
which is equal to 0 if \( N_\omega \) is simple.

Remark 4. The condition \( \lambda < \infty \) cannot be dropped. For a counter example, see Exercise 3.3.2 of [3].

Proposition 3 (Korolyuk’s theorem)  A crudely stationary, orderly point process is simple.
Proof. By Fatou’s lemma and the orderliness of $N_\omega$,

\[ E\left[ \sum_{j} : x_j(\omega) \in (0, 1], \quad m_j(\omega) \geq 2 \right] = E\left[ \liminf_{h \to 0} \frac{1}{h} \int_0^1 \mathbf{1}_{\{ N_\omega((x, x+h)) \geq 2 \}} \, dx \right] \leq \liminf_{h \to 0} \frac{1}{h} P(N_\omega((0, h]) \geq 2) = 0 \, , \]

so that with probability one, $N_\omega$ has no multiple points in $(0, 1]$. By crude stationarity, the above argument is also valid if $(0, 1]$ is replaced by $(n, n+1]$ for any integer $n$. Hence $N_\omega$ is simple.

**Proposition 4** For a crudely stationary point process $N_\omega$ with finite intensity $\lambda$, the limits

\[ \lambda_k := \lim_{h \to 0} \frac{1}{h} \int_0^1 \mathbf{1}_{\{ 1 \leq N_\omega((x, x+h)) \leq k \}} \, dx \]

exists for $k = 1, 2, \ldots$, and satisfy $\lambda_k \to \lambda$ as $k \to \infty$. Moreover for $k = 1, 2, \ldots$,

\[ \pi_k := \frac{\lambda_k - \lambda_{k-1}}{\lambda} = \lim_{h \to 0} P(N_\omega((0, h]) = k \mid N_\omega((0, h]) > 0) \, , \]

where we set $\lambda_0 := 0$.

Proof. As before, one has

\[ \frac{1}{h} \int_0^1 \mathbf{1}_{\{ 1 \leq N_\omega((x, x+h]) \leq k \}} \, dx \leq 1 + N_\omega^*(0, 1] \, , \]

and

\[ \lim_{h \to 0} \frac{1}{h} \int_0^1 \mathbf{1}_{\{ 1 \leq N_\omega((x, x+h]) \leq k \}} \, dx = \sum_{j} : x_j(\omega) \in (0, 1], \quad m_j(\omega) \leq k \, . \]

Since $\lambda = E[N_\omega^*((0, 1]]) < \infty$, we can apply the dominated convergence theorem and Lemma 2, to obtain

\[ \lambda_k = \lim_{h \to 0} \frac{1}{h} E\left[ \int_0^1 \mathbf{1}_{\{ 1 \leq N_\omega((x, x+h]) \leq k \}} \, dx \right] = E\left[ \sum_{j} : x_j(\omega) \in (0, 1], \quad m_j(\omega) \leq k \right] \, . \]

This representation of $\lambda_k$ immediately gives

\[ \lim_{k \to \infty} \lambda_k = E[\sum_{j} : x_j(\omega) \in (0, 1)] = E[N_\omega^*((0, 1])] = \lambda \, , \]

by the monotone convergence theorem. The last statement of the proposition is obvious.

**Corollary 2** For a crudely stationary point process with finite intensity, we have

\[ \lambda \sum_{k=1}^{\infty} k \pi_k = E[N_\omega((0, 1])] = m \, . \]
3. The Palm measure

Let us assume that the probability space \((\Omega, \mathcal{F}, P)\), on which our point process \(N_\omega\) is defined, is equipped with a measurable flow \(\{\theta_t\}_{t \in \mathbb{R}}\). Here a measurable flow \(\{\theta_t\}\) is, by definition, a family of bijections \(\theta_t: \Omega \to \Omega\) such that

(a) \(\theta_0\) is the identity mapping, and for any \(s, t \in \mathbb{R}\), \(\theta_s \circ \theta_t = \theta_{s+t}\) holds;
(b) the mapping \((t, \omega) \mapsto \theta_t(\omega)\) from \(\mathbb{R} \times \Omega\) into \(\Omega\) is jointly measurable with respect to \(\mathcal{B}(\mathbb{R}) \times \mathcal{F}\), where \(\mathcal{B}(\mathbb{R})\) is the Borel \(\sigma\)-algebra on \(\mathbb{R}\).

Let us further assume that the relation

\[
\int_{\mathbb{R}} N_{\theta_t \omega}(dx) \phi(x) = \int_{\mathbb{R}} N_\omega(dx) \phi(x - t) \tag{6}
\]

holds for any \(t \in \mathbb{R}\) and any continuous function \(\phi\) with compact support. If the probability measure \(P\) is \(\{\theta_t\}\)-invariant in the sense \(P \circ \theta_{-t}^{-1} = P\) for all \(t \in \mathbb{R}\), then by (6), our point process \(N_\omega\) is stationary.

Definition 5 The Palm measure of a point process \(N_\omega\) is a measure kernel \(Q(x, d\omega)\) on \(\mathbb{R} \times \Omega\) such that for any jointly measurable, non-negative function \(f(x, \omega)\), the relation

\[
\int_{\Omega} P(d\omega) \int_{\mathbb{R}} N_\omega(dx) f(x, \omega) = \int_{\mathbb{R}} \lambda(dx) \int_{\Omega} Q(x, d\omega) f(x, \omega) \tag{7}
\]

holds, where \(\lambda(dx)\) is the mean measure of \(N_\omega\) which is defined by \(\lambda(B) = \mathbb{E}[N_\omega(B)]\) for \(B \in \mathcal{B}(\mathbb{R})\) and which we assume to be finite for bounded Borel sets \(B\).

Now let \(u(t)\) be a probability density function on \(\mathbb{R}\). Define a new probability measure \(P_u\) by

\[
\int_{\Omega} P_u(d\omega) g(\omega) = \int_{\mathbb{R}} u(t) dt \int_{\Omega} P(d\omega) g(\theta_t \omega), \tag{8}
\]

where \(g(\omega)\) is an arbitrary non-negative measurable function on \(\Omega\). Then the following result holds.

Theorem 1 For any probability density \(u(t)\) on \(\mathbb{R}\), the Palm measure \(Q_u(x, d\omega)\) exists for the point process \(N_\omega\) defined on the probability space \((\Omega, \mathcal{F}, P_u)\).

Proof. Let \(f(x, \omega) \geq 0\) be jointly measurable on \(\mathbb{R} \times \Omega\). Then we can rewrite the left
hand side of (7) in the following way:
\[
\int_{\Omega} P_t(d\omega) \int_{\mathbb{R}} N_\omega(dx)f(x, \omega) = \int_{\mathbb{R}} u(t)dt \int_{\Omega} P(d\omega) \int_{\mathbb{R}} N_{\theta_t\omega}(dx)f(x, \theta_t\omega) \\
= \int_{\mathbb{R}} u(t)dt \int_{\Omega} P(d\omega) \int_{\mathbb{R}} N_\omega(dx)f(x-t, \theta_t\omega) \\
= \int_{\Omega} P(d\omega) \int_{\mathbb{R}} N_\omega(dx) \int_{\mathbb{R}} u(t)dtf(x-t, \theta_t\omega) \\
= \int_{\Omega} P(d\omega) \int_{\mathbb{R}} N_\omega(dx) \int_{\mathbb{R}} u(x-s)dsf(s, \theta_{t-s}\omega) \\
= \int_{\mathbb{R}} ds \int_{\Omega} P(d\omega) \int_{\mathbb{R}} N_\omega(dx)u(x-s)f(s, \theta_{t-s}\omega). \quad (9)
\]
At this stage, take \( f(x, \omega) = \varphi(x) \). Then (9) reduces to
\[
\int_{\mathbb{R}} \varphi(s)\lambda(ds) = \int_{\mathbb{R}} \varphi(s)\ell_u(s)ds \quad (10)
\]
with
\[
\ell_u(s) = \int_{\Omega} P(d\omega) \int_{\mathbb{R}} N_\omega(dx)u(x-s). \quad (11)
\]
If we define, for each \( s \in \mathbb{R} \), the measure \( Q_u(s, d\omega) \) on \((\Omega, \mathcal{F})\) by
\[
\int_{\Omega} Q_u(s, d\omega)g(\omega) = \frac{1}{\ell_u(s)} \int_{\mathbb{R}} P(d\omega) \int_{\mathbb{R}} N_\omega(dx)u(x-s)g(\theta_{t-s}\omega), \quad (12)
\]
then (9) takes the form of (7), and the theorem is proved.

When \( P \) is \( \{\theta_t\} \)-invariant, then we have \( P_u = P \) for any probability density \( u \) on \( \mathbb{R} \), and
\[
\ell_u(s) = \int_{\Omega} P(d\omega) \int_{\mathbb{R}} N_{\theta_u\omega}(dx)u(x) = \int_{\mathbb{R}} P(d\omega) \int_{\mathbb{R}} N_\omega(dx)u(x) =: \ell > 0
\]
is a constant. Moreover one can compute as
\[
\int_{\Omega} Q_u(s, d\omega)g(\omega) = \frac{1}{\ell} \int_{\mathbb{R}} P(d\omega) \int_{\mathbb{R}} N_{\theta_u\omega}(dx)u(x)g(\theta_u\omega) \\
= \frac{1}{\ell} \int_{\mathbb{R}} P(d\omega) \int_{\mathbb{R}} N_{\theta_u\omega}(dx)u(x)g(\theta_{t-s}\theta_u\omega)) = \frac{1}{\ell} \int_{\Omega} P(d\omega) \int_{\mathbb{R}} N_\omega(dx)u(x)g(\theta_{t-s}\omega). \\
\]
Hence if we define a measure \( \hat{P}(d\omega) \) on \((\Omega, \mathcal{F})\) by
\[
\int_{\Omega} \hat{P}(d\omega)g(\omega) = \int_{\Omega} P(d\omega) \int_{\mathbb{R}} N_\omega(dx)u(x)g(\theta_t\omega),
\]
then we get
\[
Q_u(s, d\omega) = \frac{1}{\ell}(\hat{P} \circ \theta_u)(d\omega),
\]
and (7) can be written in the form
\[
\int_{\Omega} \mathbf{P}(d\omega) \int_{\mathbb{R}} N_u(dx)f(x, \omega) = \int_{\mathbb{R}} dx \int_{\Omega} \hat{\mathbf{P}}(d\omega)f(x, \theta_x \omega),
\]  
which is the defining relation of the Palm measure in the stationary case (see [6]). (13) shows in particular that the definition of \( \hat{\mathbf{P}} \) is independent of the choice of \( u \).

Our consideration of the probability measure \( \mathbf{P}_u \) is motivated by the following observation.

**Proposition 5** The probability measure \( \mathbf{P} \) is \( \{\theta_t\} \)-invariant if and only if the following two conditions hold:

(i) \( \mathbf{P}_u = \mathbf{P} \) for any probability density function \( u(t) \) on \( \mathbb{R} \);
(ii) the set \( H \) of all bounded measurable functions \( \varphi(\omega) \) on \( \Omega \) such that \( t \mapsto \varphi(\theta_t \omega) \) is continuous for all \( \omega \in \Omega \) is dense in \( L^2(\Omega, \mathbf{P}) \).

**Proof.** The necessity of (i) is obvious. That (ii) also follows from the \( \{\theta_t\} \)-invariance of \( \mathbf{P} \) is proved in [6] (see Lemma II. 3). To prove the sufficiency of (i) and (ii), fix an arbitrary \( t_0 \in \mathbb{R} \) and take a sequence of probability density \( \{u_n\} \) so that \( u_n(t)dt \to \delta_{t_0}(dt) \) weakly. Now for any \( \varphi \in H \), \( t \mapsto \varphi(\theta_t \omega) \) is continuous and bounded by \( \|\varphi\|_{\infty} := \sup_{\Omega} |\varphi(\omega)| \). Hence we can apply the dominated convergence theorem, to get

\[
\int_{\Omega} \mathbf{P}(d\omega)\varphi(\theta_{t_0} \omega) = \int_{\Omega} \mathbf{P}(d\omega) \left( \lim_{n \to \infty} \int_{\mathbb{R}} \varphi(\theta_t \omega)u_n(t)dt \right)
= \lim_{n \to \infty} \int_{\mathbb{R}} \left( \int_{\Omega} \mathbf{P}(d\omega)\varphi(\theta_t \omega) \right)u_n(t)dt
= \lim_{n \to \infty} \int_{\Omega} \mathbf{P}_u(d\omega)\varphi(\omega) = \int_{\Omega} \mathbf{P}(d\omega)\varphi(\omega)
\]

by condition (i). But if \( H \) is dense in \( L^2(\Omega, \mathbf{P}) \), we can approximate an arbitrary bounded measurable function \( g(\omega) \) by the elements of \( H \), to obtain

\[
\int_{\Omega} \mathbf{P}(d\omega)g(\theta_{t_0} \omega) = \int_{\Omega} \mathbf{P}(d\omega)g(\omega)
\]

for any \( t_0 \in \mathbb{R} \). This sows the \( \{\theta_t\} \)-invariance of \( \mathbf{P} \).

In most cases of application, \( \Omega \) itself is a topological space with \( \mathcal{F} \) the Baire \( \sigma \)-algebra generated by that topology and \( t \mapsto \theta_t \omega \) is continuous for all \( \omega \in \Omega \). In such a case, \( H \) contains the class \( C_0(\Omega) \) of all bounded continuous functions on \( \Omega \), which is dense in \( L^2(\Omega, \mathbf{P}) \). Hence condition (ii) is not as restrictive as it may appear.
See [4] for a general treatment of stationary random measures on a topological group.

References


