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New Proofs of Some Basic Theorems on Stationary Point Processes

Nariyuki MINAMI*

Summary — We give new proofs of three basic theorems on stationary point processes on the real line — theorems of Khintchine, Korolyuk, and Dobrushin. Moreover we give a direct construction of the Palm measure for a class of point processes which includes stationary ones as special cases.

1. Introduction

The purpose of this note is to give new proofs, based on a same simple idea, to some basic theorems on stationary point processes on the real line $\mathbb{R}$, as stated in standard treatises on point processes such as Daley and Vere-Jones (see §3.3 of [3]).

To begin with, let us introduce necessary definitions and notation. By $M_p$, we denote the set of all integer-valued Radon measures on $\mathbb{R}$. Namely $M_p$ is the totality of all measures $N(dx)$ on $\mathbb{R}$ such that for any bounded Borel set $B$, $N(B)$ is a non-negative integer. Let us call any such measure a counting measure. For a counting measure $N \in M_p$, let us define

$$X(t) = \begin{cases} N((0, t]) & (t \geq 0), \\ -N((t, 0)) & (t < 0) \end{cases}$$

(1)

Then the function $X(t)$ is right-continuous, integer-valued, locally bounded and non-decreasing. Hence $X(t)$ is piecewise constant on $\mathbb{R}$ and the set $\Delta$, finite or countably infinite, of its points of discontinuity has no accumulation points other than $\pm \infty$. Thus the points in $\Delta$ can be ordered as

$$\cdots < x_{-1} < x_0 \leq 0 < x_1 < x_2 < \cdots ,$$

so that if we let $m_n := X(x_n) - X(x_n - 0)$, then $N(dx)$ can be represented as

$$N(dx) = \sum_n m_n \delta_{x_n}(dx) ,$$

(2)

where $\delta_a$ denotes the unit mass placed at $a$. Each $m_n$ is a positive integer and is called

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the multiplicity of the point \( x_\nu \). In general, either \( N ([0, \infty)) \) or \( N ((-\infty, 0)) \) can be finite, in which case either \( \{x_n\}_{n=0} \) or \( \{x_n\}_{n<0} \) is a finite sequence. If in the former [resp. latter] case \( \{x_n\}_{n=0} \) [resp. \( \{x_n\}_{n<0} \)] terminates with \( x_\nu \), then we will set \( x_n = \infty \) [resp. \( x_n = -\infty \)] for \( n > \nu \) [resp. \( n < \nu \)]. When \( m_n = 1 \) for all \( n \) such that \( x_n \neq \pm \infty \), the counting measure \( N \) is said to be simple. For each \( N \in M_p \) with representation (2), let us associate a simple counting measure \( N^* \) defined by

\[
N^*(dx) = \sum_n \delta_{x_n}(dx).
\]

In order to make \( M_p \) a measurable space, we define \( M_p \) to be the \( \sigma \)-algebra of subsets of \( M_p \) generated by all mappings of the form

\[
M_p \ni N \mapsto N((B)) \in [0, \infty]
\]

for all Borel sets \( B \subset \mathbb{R} \). Then we see that \( x_n, m_n \) and \( N^* \) are all measurable functions of \( N \), as the following lemma shows.

**Lemma 1** (i) The set

\[
C := \{N \in M_p : N((-\infty, 0]) = N([0, \infty)) = \infty\} = \{N \in M_p : x_n \text{ is finite for all } n\}
\]

belongs to \( M_p \).

(ii) For each integer \( n \), \( x_n \) and \( m_n \) are \( M_p \)-measurable functions of \( N \).

(iii) The mapping \( M_p \ni N \mapsto N^* \in M_p \) is \( M_p/M_p \)-measurable.

**Proof.** (i) The assertion is obvious from the definition of \( M_p \), since we can write

\[
C = \bigcap_{k-1}^{\infty} \bigcup_{n=1}^{\infty} \{N \in M_p : N((-n, 0]) > k, N([0, n]) > k\}.
\]

(ii) The measurability of \( x_1 \) follows from the relation

\[
\{N \in M_p : x_1 > t\} = \{N \in M_p : N((0, t)) = t\}.
\]

which holds for all \( t \geq 0 \). Now for each \( k \geq 1 \), define

\[
x_1^{(k)} := \sum_{j=1}^{\infty} \frac{1}{2^j} I_{(j-1)/2^j, j/2^j}(x_1) + \infty \cdot 1_{\{x_1 = \infty\}}.
\]

Then we see that \( x_1^{(k)} \) is measurable in \( N \) and that \( x_1^{(k)} \searrow x_1 \) as \( k \to \infty \). By the right-continuity of \( X(t) = N((0, t]) \) at \( t > 0 \), we have, as \( k \to \infty \),

\[
I_{\{x_1 < \infty\}} \cdot X(x_1^{(k)}) = \sum_{j=1}^{\infty} I_{(j-1)/2^j, j/2^j}(x_1)X\left(\frac{j}{2^k}\right) \to X(x_1) = m_1,
\]

which shows the measurability of \( m_1 \) in \( N \).

Next let \( X(t) := X(t) - X(t \wedge x_1) \). This is measurable in \( N \) for all \( t \geq 0 \), since
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\[X(t \wedge x) = X(t)\mathbf{1}_{\{x_1 \geq t\}} + X(x)\mathbf{1}_{\{x_1 < t\}}.\]

If we apply the above argument to \(\tilde{X}(t)\) instead of \(X(t)\), we can verify the measurability of \(x_2\) and \(m_2\) in \(N\), and the argument can be iterated to give the measurability of all \(x_n\) and \(m_n\).

(iii) For each \(j = 0, 1, 2, \ldots\) and \(t > 0\), the sets

\[\{N \in M_p : N^* ((0, t]) = j\} = \{N \in M_p : x_j \leq t < x_{j+1}\}\]

and

\[\{N \in M_p : N^* ((-t, 0]) = j\} = \{N \in M_p : x_{-j} \leq t < x_{-j+1}\}\]

belong to \(M_p\). Now for each \(n \geq 1\), let \(G_n\) be the class of all Borel subsets \(B\) of \([-n, n]\) such that the mapping

\[M_p \ni N \mapsto N^* (B) \in [0, \infty) \tag{5}\]

is measurable. Then \(G_n\) is seen to be a \(\lambda\)-system which contains the class of intervals

\[I := [(0, t] : 0 < t \leq t] \cup \{[-t, 0] : 0 < t \leq n\}\]

which forms a \(\pi\)-system. Hence by Dynkin’s \(\pi\)-\(\lambda\) theorem (see e.g. Durrett [2]), \(G_n\) contains all Borel subsets of \([-n, n]\). Since \(n \geq 1\) is arbitrary, and since we can write \(N^* (B) = \lim_{n \to \infty} N^* (B \cap [-n, n])\), the mapping (5) is measurable for all Borel subsets of \(\mathbb{R}\).

Remark 1. By an argument similar to (iii), it is easy to show that \(M_p\) is generated by mappings \(M_p \ni N \mapsto X(t)\) for all \(t\), where \(X(t)\) is defined in (1).

Definition 1 A point process \(N_\omega\) is a random variable defined on a probability space \((\Omega, \mathcal{F}, P)\) and taking values in the measurable space \((M_p, \mathcal{M}_p)\).

Definition 2 A point process \(N_\omega\) is said to be crudely stationary if for any bounded interval \(I\) and for any \(x \in \mathbb{R}\), \(N_\omega(I)\) and \(N_\omega(I + x)\) are identically distributed. Its mean density is the expectation value \(m := E[N_\omega ((0, 1])] \leq \infty\).

Definition 3 A point process \(N_\omega\) is said to be stationary if for any \(C \in \mathcal{M}_p\) and \(x \in \mathbb{R}\), one has the identity

\[P(N_\omega(\cdot) \in C) = P(N_\omega(x + \cdot) \in C).\]
Obviously, $N_\omega$ is crudely stationary if it is stationary.

Remark 2. By another application of $\pi$-$\lambda$ theorem, one can show without difficulty that $N_\omega$ is stationary if and only if for any finite family of Borel subsets $B_1, \ldots, B_n$ of $\mathbb{R}$, and of non-negative integers $k_1, \ldots, k_n$, the identity

$$P(N_\omega(B_i) = k_i, i = 1, \ldots, n) = P(N_\omega(x + B_i) = k_i, i = 1, \ldots, n)$$

holds for any $x \in \mathbb{R}$.

2. Basic theorems and their proofs

Our argument is based on the following lemma, which is an immediate consequence of Definition 2.

Lemma 2 Let the point process $N_\omega$ be crudely stationary. Then for any bounded interval $I$ and for any non-negative integer $k$,

$$P(N_\omega(I) = k) = \int_0^1 P(N_\omega(x + I) = k)dx = E\left[\int_0^1 1_{\{N_{\omega(\cdot, x + I)} = k\}}dx\right].$$

Proposition 1 (Khintchine’s theorem) For any crudely stationary point process $N_\omega$, the limit

$$\lambda := \lim_{K, h \to \infty} \frac{1}{K} P(N_\omega((0, h]) > 0)$$

exists and satisfies $\lambda \leq m$. $\lambda$ is called the intensity of the point process $N_\omega$.

Proof. Let $N_\omega$ be represented as (2) and define the point process $N_\omega^*$ by (3). If we set $\nu(\omega) := N_\omega^*(0, 1]$, it satisfies $x_{\nu(\omega)}(\omega) \leq 1 < x_{\nu(\omega)+1}(\omega)$. Obviously we have

$$\{x \in (0, 1] : N_\omega([x, x + h]) > 0\} = (0, 1] \cap \bigcup_{j=1}^{\nu(\omega)} [x_j(\omega) - h, x_j(\omega))$$

$$= (0, 1] \cap \bigcup_{j=1}^{\nu(\omega)+1} J_j^\omega(h) = \sum_{j=1}^{\nu(\omega)+1} ([0, 1] \cap (J_j^\omega(h) \setminus J_{j-1}^\omega(h))),$$

where we have set $J_j^\omega(h) := [x_j(\omega) - h, x_j(\omega))$ and $J_0 = \emptyset$. Hence

$$\frac{1}{h} \int_0^1 1_{\{N_\omega([x, x + h]) > 0\}}dx = \frac{1}{h} \sum_{j=1}^{\nu(\omega)+1} [0, 1] \cap (J_j^\omega(h) \setminus J_{j-1}^\omega(h))$$

$$= \sum_{j=1}^{\nu(\omega)+1} \frac{1}{h} \{(1 \wedge x_j(\omega)) - (0 \vee x_{j-1}(\omega) \vee (x_j(\omega) - h)\},$$
where for a Borel subset \( B \) of \( \mathbb{R} \), \(|B|\) denotes its Lebesgue measure and for a real number \( a \), \( a^+ := a \vee 0 = \max\{a, 0\} \) denotes its positive part. Now it is easy to see that for \( 1 \leq j \leq \nu(\omega) \),

\[
\frac{1}{h} \{ 1 \wedge x_j(\omega) - (0 \lor x_{j-1}(\omega) \lor (x_j(\omega) - h) \} \nearrow 1
\]

as \( h \searrow 0 \), and that for \( j = \nu(\omega) + 1 \),

\[
\frac{1}{h} \{ (1 \wedge x_{\nu(\omega)+1}(\omega)) - (0 \lor x_{\nu(\omega)}(\omega) \lor (x_{\nu(\omega)+1}(\omega) - h) \}
\]

is bounded by 1 and tends to 0 as \( h \searrow 0 \). Thus we can apply the monotone convergence theorem, the dominated convergence theorem and Lemma 2, to obtain

\[
\frac{1}{h} \mathbb{P}(N_\omega((0, h]) > 0) = \mathbb{E} \left[ \frac{1}{h} \sum_{j=1}^{\nu(\omega)+1} \mathbb{1}_{[0, 1]}(J_j^\omega \setminus J_{j-1}^\omega(h)) \right] \rightarrow \mathbb{E} \left[ \sum_{j=1}^{\nu(\omega)} 1 \right] = \mathbb{E} [N_\omega^*(0, 1)],
\]

as \( h \searrow 0 \). Thus the desired limit \( \lambda \) exists and is equal to \( \mathbb{E}[N_\omega^*(0, 1)] \). Clearly it satisfies the inequality \( \lambda \leq \mathbb{E}[N_\omega((0, 1])] \). Clearly it satisfies

**Corollary 1** If \( N_\omega \) is simple, then \( \lambda = m \). When \( m < \infty \), the converse is also true.

**Proof.** \( N_\omega \) is simple if and only if \( N_\omega^* = N_\omega \) almost surely, which obviously implies \( \lambda = m \). On the other hand, if \( \lambda = m < \infty \), then

\[
\mathbb{E}[N_\omega((0, 1])] - N_\omega^*((0, 1]]) = m - \lambda = 0.
\]

But \( N_\omega((0, 1]) - N_\omega^*((0, 1]) \geq 0 \) in general, so that \( N_\omega((0, 1]) = N_\omega^*((0, 1]) \) almost surely. The same argument is valid if the interval \( (0, 1] \) is replaced by \( (n, n + 1] \), so that \( N_\omega((n, n + 1]) = N_\omega^*((n, n + 1]) \) almost surely for all integers \( n \), and the simplicity of \( N_\omega \) follows.

**Remark 3.** In the treatise by Daley and Vere-Jones [3], for example, Proposition 1 is proved in the following way: If we define \( \varphi(h) := \mathbb{P}(N_\omega((0, h]) > 0) \), then by the crude stationarity, we have for any positive \( h_1 \) and \( h_2 \),

\[
\varphi(h_1 + h_2) = \mathbb{P}(N_\omega((0, h_1 + h_2]) > 0) = \mathbb{P}(N_\omega((0, h_1]) + N_\omega((h_1, h_1 + h_2]) > 0) \\
\leq \mathbb{P}(N_\omega((0, h_1]) > 0) + \mathbb{P}(N_\omega((h_1, h_1 + h_2]) > 0) = \varphi(h_1) + \varphi(h_2),
\]

so that \( \varphi(h) \) is a sub-additive function defined on \([0, \infty)\) satisfying \( \varphi(0) = 0 \). To show the existence of the intensity \( \lambda \), it suffices to apply the following well known lemma.
Lemma 3 Let \( g(x) \) be a sub-additive function defined on \([0, \infty)\) such that \( g(0) = 0 \). Then one has

\[
\lim_{x \to 0} \frac{g(x)}{x} = \sup_{x > 0} \frac{g(x)}{x} \leq \infty.
\]

However, this argument does not provide the representation \( \lambda = \mathbb{E}[N^* \omega((0, 1]) \), so that the proof of Corollary 1 requires some extra work. Our proof above is closer to that of Leadbetter [5]. See also Chung [1].

Definition 4 A crudely stationary point process \( N_\omega \) is said to be orderly when

\[
\mathbb{P}(N_\omega((0, h]) \geq 2) = o(h) \quad (h \searrow 0).
\]

Proposition 2 (Dobrushin’s theorem) If a crudely stationary point process \( N_\omega \) is simple and if \( \lambda < \infty \), then \( N_\omega \) is orderly.

Proof. By Lemma 2, we can write

\[
\mathbb{P}(N_\omega((0, h]) \geq 2) = \mathbb{E}\left[ \int_0^h 1_{\{N_\omega((x, x+h]) \geq 2\}} dx \right].
\]

As can be seen from the proof of Proposition 1, we have

\[
\frac{1}{h} \int_0^h 1_{\{N_\omega((x, x+h]) \geq 2\}} dx \leq \frac{1}{h} \int_0^h 1_{\{N_\omega((x, x+h]) > q\}} dx
\]

\[
= \sum_{j=1}^{\eta(0)} \frac{1}{h} (\{0, 1\} \cap (J_j(h) \setminus J_{j-1}(h))) + \frac{1}{h} \{0, 1\} \cap (J_{\eta(0)+1}(h) \setminus J_{\eta(0)}(h))
\]

\[
\leq N^*_\omega((0, 1]) + 1,
\]

and

\[
\lim_{h \searrow 0} \frac{1}{h} \int_0^h 1_{\{N_\omega((x, x+h]) \geq 2\}} dx = \mathbb{E}\{j : x_j(\omega) \in (0, 1], \eta_j(\omega) \geq 2\}.
\]

Since \( \mathbb{E}[N^*_\omega((0, 1])] = \lambda < \infty \), we can apply the dominated convergence theorem, to obtain

\[
\lim_{h \searrow 0} \frac{1}{h} \mathbb{P}(N_\omega((0, h]) \geq 2) = \mathbb{E}\{j : x_j(\omega) \in (0, 1], \eta_j(\omega) \geq 2\},
\]

which is equal to 0 if \( N_\omega \) is simple.

Remark 4. The condition \( \lambda < \infty \) cannot be dropped. For a counter example, see Exercise 3.3.2 of [3].

Proposition 3 (Korolyuk’s theorem) A crudely stationary, orderly point process is simple.
Proof. By Fatou's lemma and the orderliness of $N_w$, 

$$E[\{ j : x_j(\omega) \in (0, 1], \, m_j(\omega) \geq 2 \}] = E\left[ \liminf_{h \searrow 0} \frac{1}{h} \int_0^1 1_{\{ N_w(\xi, x, h) \geq 2 \}} \, dx \right] \leq \liminf_{h \searrow 0} \frac{1}{h} P(\{ N_w((0, h]) \geq 2 \}) = 0,$$

so that with probability one, $N_w$ has no multiple points in $(0, 1]$. By crude stationarity, the above argument is also valid if $(0, 1]$ is replaced by $(n, n+1]$ for any integer $n$. Hence $N_w$ is simple.

**Proposition 4** For a crudely stationary point process $N_w$ with finite intensity $\lambda$, the limits

$$\lambda_k := \lim_{h \searrow 0} \frac{1}{h} P(1 \leq N_w((0, h]) \leq k)$$

exists for $k = 1, 2, \ldots$, and satisfy $\lambda_k \to \lambda$ as $k \to \infty$. Moreover for $k = 1, 2, \ldots$,

$$\pi_k := \frac{\lambda_k - \lambda_{k-1}}{\lambda} = \lim_{h \searrow 0} P(\{ N_w((0, h]) = k \, | \, N_w((0, h]) > 0 \},$$

where we set $\lambda_0 := 0$.

Proof. As before, one has

$$\frac{1}{h} \int_0^1 1_{\{ 1 \leq N_w(\xi, x, h) \leq k \}} \, dx \leq 1 + N_w^{\ast}(0, 1],$$

and

$$\lim_{h \searrow 0} \frac{1}{h} \int_0^1 1_{\{ 1 \leq N_w(\xi, x, h) \leq k \}} \, dx = \pi_k [1 : x_j(\omega) \in (0, 1], \, m_j(\omega) \leq k].$$

Since $\lambda = E[N_w^{\ast}(0, 1)] < \infty$, we can apply the dominated convergence theorem and Lemma 2, to obtain

$$\lambda_k = \lim_{h \searrow 0} \frac{1}{h} E\left[ \int_0^1 1_{\{ 1 \leq N_w(\xi, x, h) \leq k \}} \, dx \right] = E[\{ j : x_j(\omega) \in (0, 1], \, m_j(\omega) \leq k \}].$$

This representation of $\lambda_k$ immediately gives

$$\lim_{k \to \infty} \lambda_k = E[\{ j : x_j(\omega) \in (0, 1]\}] = E[N^{\ast}(0, 1)] = \lambda,$$

by the monotone convergence theorem. The last statement of the proposition is obvious.

**Corollary 2** For a crudely stationary point process with finite intensity, we have

$$\lambda \sum_{k=1}^{\infty} k \pi_k = E[N_w((0, 1])] = m.$$
3. The Palm measure

Let us assume that the probability space \((\Omega, \mathcal{F}, P)\), on which our point process \(N_\omega\) is defined, is equipped with a measurable flow \(\{\theta_t\}_{t \in \mathbb{R}}\). Here a measurable flow \(\{\theta_t\}\) is, by definition, a family of bijections \(\theta_t : \Omega \to \Omega\) such that

(a) \(\theta_0\) is the identity mapping, and for any \(s, t \in \mathbb{R}\), \(\theta_s \circ \theta_t = \theta_{s+t}\) holds;
(b) the mapping \((t, \omega) \mapsto \theta_t(\omega)\) from \(\mathbb{R} \times \Omega\) into \(\Omega\) is jointly measurable with respect to \(\mathcal{B}(\mathbb{R}) \times \mathcal{F}\), where \(\mathcal{B}(\mathbb{R})\) is the Borel \(\sigma\)-algebra on \(\mathbb{R}\).

Let us further assume that the relation

\[
\int_{\mathbb{R}} N_{\theta_t}(dx) \phi(x) = \int_{\mathbb{R}} N_\omega(dx) \phi(x-t) \tag{6}
\]

holds for any \(t \in \mathbb{R}\) and any continuous function \(\phi\) with compact support. If the probability measure \(P\) is \(\{\theta_t\}\)-invariant in the sense \(P \circ \theta_t^{-1} = P\) for all \(t \in \mathbb{R}\), then by (6), our point process \(N_\omega\) is stationary.

**Definition 5** The Palm measure of a point process \(N_\omega\) \((dx)\) is a measure kernel \(Q(x, d\omega)\) on \(\mathbb{R} \times \Omega\) such that for any jointly measurable, non-negative function \(f(x, \omega)\), the relation

\[
\int_{\Omega} P(d\omega) \int_{\mathbb{R}} N_\omega(dx) f(x, \omega) = \int_{\mathbb{R}} \lambda(dx) \int_{\Omega} Q(x, d\omega) f(x, \omega) \tag{7}
\]

holds, where \(\lambda(dx)\) is the mean measure of \(N_\omega\) which is defined by \(\lambda(B) = E[N_\omega(B)]\) for \(B \in \mathcal{B}(\mathbb{R})\) and which we assume to be finite for bounded Borel sets \(B\).

Now let \(u(t)\) be a probability density function on \(\mathbb{R}\). Define a new probability measure \(P_u\) by

\[
\int_{\Omega} P_u(d\omega) g(\omega) = \int_{\mathbb{R}} u(t) dt \left( \int_{\Omega} P(d\omega) g(\theta_t \omega) \right) \tag{8}
\]

where \(g(\omega)\) is an arbitrary non-negative measurable function on \(\Omega\). Then the following result holds.

**Theorem 1** For any probability density \(u(t)\) on \(\mathbb{R}\), the Palm measure \(Q_u(x, d\omega)\) exists for the point process \(N_\omega\) defined on the probability space \((\Omega, \mathcal{F}, P_u)\).

**Proof.** Let \(f(x, \omega) \geq 0\) be jointly measurable on \(\mathbb{R} \times \Omega\). Then we can rewrite the left
hand side of (7) in the following way:

\[
\int_{\Omega} P_\epsilon(d\omega) \int_{\mathbb{R}} N_\omega(dx)f(x, \omega) = \int_{\mathbb{R}} u(t)dt \int_{\Omega} P(d\omega) \int_{\mathbb{R}} N_{\theta, \omega}(dx)f(x, \theta, \omega)
\]

\[
= \int_{\mathbb{R}} u(t)dt \int_{\Omega} P(d\omega) \int_{\mathbb{R}} N_\omega(dx)f(x - t, \theta, \omega)
\]

\[
= \int_{\Omega} P(d\omega) \int_{\mathbb{R}} N_\omega(dx) \int_{\mathbb{R}} u(t)dtf(x - t, \theta, \omega)
\]

\[
= \int_{\Omega} P(d\omega) \int_{\mathbb{R}} N_\omega(dx) \int_{\mathbb{R}} u(x - s)dsf(s, \theta_{x-s}, \omega)
\]

\[
= \int_{\mathbb{R}} ds \int_{\Omega} P(d\omega) \int_{\mathbb{R}} N_\omega(dx)u(x - s)f(s, \theta_{x-s}, \omega) . \tag{9}
\]

At this stage, take \(f(x, \omega) = \varphi(x)\). Then (9) reduces to

\[
\int_{\mathbb{R}} \varphi(s)\lambda(ds) = \int_{\mathbb{R}} \varphi(s)\ell_u(s)ds
\]

with

\[
\ell_u(s) = \int_{\Omega} P(d\omega) \int_{\mathbb{R}} N_\omega(dx)u(x - s) . \tag{11}
\]

If we define, for each \(s \in \mathbb{R}\), the measure \(Q_u(s, d\omega)\) on \((\Omega, F)\) by

\[
\int_{\Omega} Q_u(s, d\omega) g(\omega) = \frac{1_{\{0,\omega\}[(\ell_u(s))]}(\omega)}{\ell_u(s)} \int_{\Omega} P(d\omega) \int_{\mathbb{R}} N_\omega(dx)u(x - s)g(\theta_{x-s}\omega) , \tag{12}
\]

then (9) takes the form of (7), and the theorem is proved.

When \(P\) is \(\{\theta_s\}\)-invariant, then we have \(P_\epsilon = P\) for any probability density \(u\) on \(\mathbb{R}\), and

\[
\ell_u(s) = \int_{\Omega} P(d\omega) \int_{\mathbb{R}} N_{\theta_s\omega}(dx)u(x) = \int_{\Omega} P(d\omega) \int_{\mathbb{R}} N_\omega(dx)u(x) =: \ell > 0
\]

is a constant. Moreover one can compute as

\[
\int_{\Omega} Q_u(s, d\omega) g(\omega) = \frac{1}{\ell} \int_{\Omega} P(d\omega) \int_{\mathbb{R}} N_{\theta_s\omega}(dx)u(x)g(\theta_{s}\omega)
\]

\[
= \frac{1}{\ell} \int_{\Omega} P(d\omega) \int_{\mathbb{R}} N_{\theta_s\omega}(dx)u(x)g(\theta_{s-x}\omega)) = \frac{1}{\ell} \int_{\Omega} P(d\omega) \int_{\mathbb{R}} N_\omega(dx)u(x)g(\theta_{s-x}\omega) .
\]

Hence if we define a measure \(\hat{P}(d\omega)\) on \((\Omega, F)\) by

\[
\int_{\Omega} \hat{P}(d\omega) g(\omega) = \int_{\Omega} P(d\omega) \int_{\mathbb{R}} N_\omega(dx)u(x)g(\theta_x\omega) ,
\]

then we get

\[
Q_u(s, d\omega) = \frac{1}{\ell}[\hat{P} \circ \theta_s](d\omega) ,
\]

and (7) can be written in the form
\[ \int_{\Omega} \mathbb{P}(d\omega) \int_{\mathbb{R}} N_n(dx) f(x, \omega) = \int_{\mathbb{R}} dx \int_{\Omega} \hat{\mathbb{P}}(d\omega) f(x, \theta_n \omega), \]  

which is the defining relation of the Palm measure in the stationary case (see [6]). (13) shows in particular that the definition of \( \hat{\mathbb{P}} \) is independent of the choice of \( u \).

Our consideration of the probability measure \( \mathbb{P}_u \) is motivated by the following observation.

**Proposition 5** The probability measure \( \mathbb{P} \) is \( \{\theta_t\}\)-invariant if and only if the following two conditions hold:

(i) \( \mathbb{P}_u = \mathbb{P} \) for any probability density function \( u(t) \) on \( \mathbb{R} \);

(ii) the set \( H \) of all bounded measurable functions \( \varphi(\omega) \) on \( \Omega \) such that \( t \mapsto \varphi(\theta_t \omega) \) is continuous for all \( \omega \in \Omega \) is dense in \( L^2(\Omega, \mathbb{P}) \).

**Proof.** The necessity of (i) is obvious. That (ii) also follows from the \( \{\theta_t\}\)-invariance of \( \mathbb{P} \) is proved in [6] (see Lemma II. 3). To prove the sufficiency of (i) and (ii), fix an arbitrary \( t_0 \in \mathbb{R} \) and take a sequence of probability density \( \{u_n\} \) so that \( u_n(t)dt \to \delta_{t_0}(dt) \) weakly. Now for any \( \varphi \in H, t \mapsto \varphi(\theta_t \omega) \) is continuous and bounded by \( \|\varphi\|_{\infty} := \sup_{\Omega} |\varphi(\omega)| \). Hence we can apply the dominated convergence theorem, to get

\[
\int_{\Omega} \mathbb{P}(d\omega) \varphi(\theta_{t_0} \omega) = \int_{\Omega} \mathbb{P}(d\omega) \left( \lim_{n \to \infty} \int_{\mathbb{R}} \mathbb{P}(d\omega) \varphi(\theta_t \omega) u_n(t) dt \right)
= \lim_{n \to \infty} \int_{\mathbb{R}} \left( \int_{\Omega} \mathbb{P}(d\omega) \varphi(\theta_t \omega) \right) u_n(t) dt
= \lim_{n \to \infty} \int_{\Omega} \mathbb{P}_n(d\omega) \varphi(\omega) = \int_{\Omega} \mathbb{P}(d\omega) \varphi(\omega)
\]

by condition (i). But if \( H \) is dense in \( L^2(\Omega, \mathbb{P}) \), we can approximate an arbitrary bounded measurable function \( g(\omega) \) by the elements of \( H \), to obtain

\[
\int_{\Omega} \mathbb{P}(d\omega) g(\theta_{t_0} \omega) = \int_{\Omega} \mathbb{P}(d\omega) g(\omega)
\]

for any \( t_0 \in \mathbb{R} \). This sows the \( \{\theta_t\}\)-invariance of \( \mathbb{P} \).

In most cases of application, \( \Omega \) itself is a topological space with \( \mathcal{F} \) the Baire \( \sigma \)-algebra generated by that topology and \( t \mapsto \theta_t \omega \) is continuous for all \( \omega \in \Omega \). In such a case, \( H \) contains the class \( C_0(\Omega) \) of all bounded continuous functions on \( \Omega \), which is dense in \( L^2(\Omega, \mathbb{P}) \). Hence condition (ii) is not as restrictive as it may appear.
See [4] for a general treatment of stationary random measures on a topological group.

References