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AXIOMATIC CHARACTERIZATIONS OF CONTEST SUCCESS FUNCTIONS AND EXPLORATION OF EQUILIBRIUM

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Abstract: Skaperdas (1996) characterized the Tullock (1980)-Hirschleifer (1989) contest success functions (CSFs),which stipulate the winning probabilities of the contestants, using respectively the scale invariance and translation invariance axioms. This paper first characterizes the entire family of CSFs that fulfils an μ-independence axiom, a convex mixture of the two invariance axioms, where $0 \leq \mu \leq 1$ is a value judgment parameter. This family contains the Tullock and Hirschleifer CSFs as special cases. Next, we consider two axioms related to ranking, scale consistency and translation consistency, and characterize the respective classes of CSFs. It has been demonstrated that scale consistency and translation consistency, in the presence of other axioms, characterize the same functional forms identified by scale and translation invariances respectively. Finally, we define an intermediate μ-consistency condition and classify all CSFs satisfying the same. We also explore the possibility of existence of Nash equilibrium of the contest game satisfying μ-independence and the corresponding equilibrium efforts.

Key words: Contest, non-cooperative game, success function, invariance axioms, ranking axioms, characterizations, Nash equilibrium.

JEL Classification Number: C70, D72, D74.

1. INTRODUCTION

A contest refers to a non-cooperative game in which two or more participants contend for a prize. Models of contest have been employed extensively to analyse a variety of phenomena like rent seeking (Tullock 1980, Nitzan 1991, Baye et al. 2005, Amegashie,

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2006), conflict (Hirshleifer 1989, Skaperdas 1992), polarization (Esteban and Ray 2011, Chakravarty, 2015), electoral candidacy (Snyder 1989, Skaperdas and Grofman 1995), sporting tournament (Szymanzki 2003), provision of public goods (Kolmar and Wagener 2012) and reward structure in firms (Rosen 1985)[1]. In a contest, agents make irretrievable investments, which depending on the situation; can be money, effort or any other valuable resource.

Essential to the notion of a contest is a Contest Success Function (CSF), which specifies a contestant's probability of winning the contest and obtaining a prize. An increase in each contestant's outlay increases his chances of winning the contest and reduces his opponents' chances. In a highly interesting contribution, Skaperdas (1996) characterized this probability for any contestant as the ratio between the level of effective investment made by the contestant and the sum of effective investments across all the contestants. The effective investment of a contestant can be interpreted as the output determined by his effort, which may be regarded as his input in the contest. It is assumed to be an increasing and positive valued function of effort.This is the basic structure of Skaperdas (1996).

Using his basic structure, Skaperdas (1996) also developed axiomatic characterizations of the Tullock (1980)-Hirschleifer(1989) functional forms of CSFs. One of the axioms employed by Skaperdas (1996) is an anonymity principle which demands that a contestant's probability of success depends only on his outlays. Thus, the agents are not distinguished by any characteristic other than their outlays. Clark and Riis (1998) broadened the Skaperdas (1996) framework by allowing the contestants to differ with respect to their contest-related personal characteristics. Rai and Sarin (2009) generalized the characterizations of Skaperdas (1996) to the situation where agents can have investments that are of multiple types in nature. Münster (2009) extended the Skaperdas (1996) and Clark and Riis (1998) characterizations to contests between groups. Arbatskaya and Mialon (2010) developed a model for a multi-armed contest and characterized the CSF axiomatically in this context.

The basic structure of Skaperdas (1996) points out how to derive general CSFs that satisfy five basic axioms, namely, Efficiency, Monotonicity, Anonymity, Consistency and Independence of Outsiders' Efforts (see Section 2). However, without invoking any further condition, characterizations of the general consistent class of contest success functions will not yield any specific form of CSFs. Skaperdas (1996) invokes two alternative axioms of invariance. The first axiom, the scale invariance postulate, demands that an equi-proportionate change in the efforts of all the agents will keep the winning probabilities unchanged. In contrast, the second axiom, which is known as the translation invariance postulate, requires invariance of winning probabilities under equal translational changes in the efforts of all the agents. The underlying effective investment functions turn out to be of power function and logit function types respectively.

[1] The literature has been surveyed by Nitzan (1994), Corchon (2007), Konrad (2009) and Skaperdas and Garfinkel (2012). See also Dixit (1987) for a general discussion.

A natural generalization of the scale and translation invariance axioms is a μ-independence condition, which stipulates that a convex mixture of an equi-proportionate change and an equal absolute change in the efforts should keep winning probabilities unchanged, where $0 \leq \mu \leq 1$ is a parameter that represents a policy evaluator's judgment on invariance of winning probabilities (See Section 2 for more discussion.) One objective of this paper is to characterize the entire class of CSFs that satisfies this generalized invariance concept. It is explicitly shown that the Tullock and Hirschleifer functional forms characterized by Skaperdas (1996) become particular cases of the CSF that fulfils the μ-independence postulate.

Given two contests CI and CII, investors may be interested in ranking them in terms of their probabilities of winning. It is natural that the contest in which winning probabilities are not lower will be preferred. It is also natural that the choice remains invariant under any increasing transformation of probabilities, since an increasing transformation of probabilities generates again a probability distribution. Evidently, this is a general postulate. However, in order to pin down some specific functional forms of CSFs, one needs to impose some value judgement postulate. In fact, in the last few years attempts have been made to provide foundations of commonly used CSFs[2]. One such postulate that ensures ranking property of CSFs is the scale consistency axiom, which says that if all the agents are participating in two contests and for some agents the probabilities of winning one contest are less than or equal to that of winning the other, then an equi-proportionate change in the efforts of the agents in both contests will not alter the agents' ranking of chances of winning the contests. To understand this, suppose the investments are measured in money units, say euro. Then suppose some individuals' chances of winning CI are more than that of CII. Now, if investments are converted into dollars from euro, the inequality between chances of winning CI and CII should not alter. Scale consistency demands this condition.Note that since the sum of probabilities of winning a contest across the agents is one, if for some agents the probabilities of winning one contest over another are lower, then there will be at least one agent for whom the reverse inequality for probabilities of winning the contests will hold. CSFs satisfying scale invariance are definitely scale consistent.

Likewise, we can have a translation consistency axiom, which specifies that inequality between winning probabilities for two contests should remain invariant under equal absolute changes in all the efforts. Translation invariance implies translation consistency. However, as we will demonstrate, if the number of contestants is only 2, there can be CSFs that satisfy scale (respectively, translation) consistency but not scale (respectively, translation) invariance.

A second objective of the paper is to axiomatize the classes of CSFs that are scale and translation consistent respectively. It is fairly interesting to observe that if the number of contestants is greater than 2, the Tullock and the Hirschleifer CSFs turn out to be the

[2] See the survey papers referred to in footnote 1.Some authors have also attempted to develop econometric estimation of several CSFs. (See Jia and Skaperdas, 2011 and Jia, Skaperdas and Vaidya, 2013, for detailed discussions.)

only CSFs that verify scale and translation consistency axioms respectively. Thus, both the Tullock and the Hirschleifer CSFs can be characterized by ranking axioms. This is another attractive feature of our paper.

Then, we define an intermediate μ-consistency condition, which may be viewed as the ranking counterpart of μ-independence. Alternatively, it can be seen as a convex mixture of translation consistency and scale consistency. We demonstrate that if the number of contestants is greater than 2, the only class of intermediate μ-consistent CSFs is necessarily μ-independent. We then analyze the likelihood of occurrence of Nash equilibrium for the CSF derived using this generalized invariance concept. It is known that the Tullock CSF has Nash equilibrium in pure strategies and the Hirschleifer CSF has no Nash equilibrium in pure strategies.

In a recent contribution, using a model of military conflict, Hwang (2012) showed how the underlying CSFs influence governments' decisions on military spending and policies such as declaration of the war and settlements. He provided an axiomatization of CSFs that includes the Tullock and Hirshleifer CSFs as special cases. Essential to this characterization is the elasticity of augmentation which measures the extent to which one country requires to increase its existing resources to keep its winning probabilities unchanged in response to an increase in its opponent's resources. One of his postulates used constancy of this elasticity of augmentation.

We demonstrate that the CSF satisfying the generalized invariance axiom has a unique Nash equilibrium in pure strategies and this equilibrium can as well be a corner solution in a pure μ-independence situation, which coincides neither with the relative nor with the absolute invariance case. It may be noted that the existence of a Nash equilibrium as a corner solution is not possible for the Tullock CSF.

The paper is organized as follows. The formal framework is presented in Section 2. Section 3 deals with equilibrium analysis. Finally, Section 4 concludes.

2. THE FORMAL FRAMEWORK

Let $N = \{1, 2, \ldots, n\}$ be a set of agents participating in a contest and let y_i stand for effort or investment of agent $i \in N$ in the contest. It is assumed at the outset that$n \geq 3$. (In fact, this is required in one characterization of Skaperdas (1996)). We denote the vector of investments $(y_1, y_2, \ldots, y_n) \in [0, \infty)^n$ by $\boldsymbol{y}$, where $[0, \infty)^n$ is the $n-$fold Cartesian product of $[0, \infty)$. The success of any contestant is probabilistic. For any $\boldsymbol{y} \in [0, \infty)^n$, each contestant$i$'s probability of winning the contest is denoted by$p^i(\boldsymbol{y})$. Evidently, $p^i : [0, \infty)^n \to [0, 1]$. The non-negative function p is called the Contest Success Function (CSF).

The following axioms for a CSF have been suggested by Skaperdas (1996).

(A1) (Efficiency) For all $\boldsymbol{y} \in [0, \infty)^n$, $\sum_{i=1}^{n} p^i(\boldsymbol{y}) = 1$ and if $y_i > 0$ then $p^i(\boldsymbol{y}) > 0$.

(A2) (Monotonicity) For all $\boldsymbol{y} \in [0, \infty)^n$, $p^i(\boldsymbol{y})$ is strictly increasing in y_i and strictly decreasing in y_j for all $j \neq i$.

(A3) (Anonymity) For all $\boldsymbol{y} \in [0, \infty)^n$, any permutation $\pi : N \to N, p^{\pi(i)}(\boldsymbol{y}) = p^i\left(y_{\pi_1}, y_{\pi_2}, \ldots, y_{\pi_n}\right)$.

(A4) (Proportionality) For all $y \in [0, \infty)^n$, for all $M \subseteq N$with at least two elements, the probability of success of agent $i \in M$in a contest among the members of $M$is$p^i_M(y) = \frac{p^i(\boldsymbol{y})}{\sum_{j \cup M} p^j(\boldsymbol{y})}$; provided that there is at least one $j \in M$ such that $p^j(\boldsymbol{y}) > 0$, where $y \neq 01^n$, where $\mathbf{1}^n$ is the n-coordinated vector of ones.

(A5) (Independence of Outsiders' Efforts) For all$\boldsymbol{y} \in [0, \infty)^n$, $p^i_M(\boldsymbol{y})$ is independent of the efforts of the players not included in the subset $M \subseteq N$or $p^i_m(\boldsymbol{y})$ can be written as$p^i_M\left(\boldsymbol{y}^M\right)$, where $y^M = \left(y_j : j \in M\right)$.

(A5′) For all $\boldsymbol{y} \in [0, \infty)^n$, $p^i(\boldsymbol{y}) = \dfrac{f(y_i)}{\sum_{j \in N} f\left(y_j\right)}$ for all $i \in N$and $p^i_M(\boldsymbol{y}) = \dfrac{f(y_i)}{\sum_{j \in M} f\left(y_j\right)}$ for all$i \in M\ (\subseteq N)$, provided that there is $j \in M$ with $y_j > 0$, where $f : [0, \infty) \to [0, \infty)$is strictly increasing in its argument and $\boldsymbol{y} \neq 01^n$.

(A1) states that the sum of winning probabilities across the participants in a contest is 1 and if some participant's outlay is positive he has a positive chance of winning the contest. (A2) says that a participant's probability of success is strictly increasing in his own effort but strictly decreasing in the efforts of the other participants. According to (A3), the probability of success remains invariant under any reordering of the participants. This anonymity condition demands that any characteristic other than individual outlays is irrelevant to the determination of success probabilities. The consistency condition (A4) says that for any subgroup of participants, the probabilities of success of the members of the subgroup are the conditional probabilities obtained by restricting the original probability distribution to the subgroup. For (A4) to be well-defined, it is necessary to assume, under (A1), that $y \neq 01^n$. Otherwise the denominator on the right hand side of $p^i_M(\boldsymbol{y})$ may vanish. (A5) means that for any subgroup of participants, the success probabilities are independent of the outlays of the participants who are not members of the subgroup. Finally, (A5′) provides a particular specification of the winning probabilities using a positive valued strictly increasing function of efforts. We can refer to $f(y_i)$ as the effective investment made by contestant i. Strict increasingness of f reflects the view that an increase in the actual investment strictly increases effective investment. Skaperdas (1996) demonstrated that (A1)–(A5) hold simultaneously if and only if the CSF is of the form specified in (A5′).Since our characterizations employ the basic axioms (A1)–(A5), we will deal with the general form given by (A5′).

Note that the expression of $p^i_M(\boldsymbol{y})$ given by (A5′) is undefined at all those points where

$$\sum_{j \in M} f\left(y_j\right) = 0\,. \tag{1}$$

By strict increasingness of f it follows that $f(y_j) > f(0) \geq 0$ whenever $y_j \in (0, \infty)$. Thus, (1) is an impossibility if there is $j \in M$ such that $y_j > 0$. Moreover, given the structure of the function f, $p^i(\boldsymbol{y})$ will be defined and continuous everywhere on $[0, \infty)^n \backslash \{(0, \ldots, 0)\}$. Thus, if $f(0) = 0$, the domain of the CSF defined in Skaperdas (1996) excludes the origin. (See Section 3 for a discussion on Corchon's (2007) suggestion along this line.)

As stated in the Introduction, some additional axiom(s) have to be invoked in order to identify specific functional forms of CSFs. Skaperdas (1996) imposed the following axioms:

(A6) (Scale invariance) For all $\boldsymbol{y} \in [0, \infty)^n$, $p^i(\boldsymbol{y}) = p^i(\lambda \boldsymbol{y})$ for all $\lambda > 0$ and for all $i \in N$.

(A7) (Translation invariance) For all $\boldsymbol{y} \in [0, \infty)^n$, $p^i(\boldsymbol{y}) = p^i(\boldsymbol{y} + c1^n)$, where 1^n is the n-coordinated vector of ones and c is a scalar such that $y_i + c \geq 0$ for all $i \in N$.

The scale invariance axiom (A6) is a homogeneity condition, which says that proportional changes in the efforts of all the contestants do not change the winning probabilities. In contrast, (A7) is a translation invariance axiom, which demands that winning probabilities remain unchanged when all the efforts are augmented or diminished by the same absolute quantity.

It has been shown in Skaperdas (1996) that a CSF defined (and continuous) on $[0, \infty)^n \backslash \{(0, \ldots, 0)\}$ satisfies (A1)–(A6) if and only if it is of the power function type, that is, of the form $p^i(\boldsymbol{y}) = \dfrac{y_i^\delta}{\sum_{j \in N} y_j^\delta}$, where $\delta > 0$ is a constant. This is the Tullock (1980) form of CSF. It has a Nash equilibrium in pure strategies for $\delta \in (0, 1]$. The particular case $\delta = 1$ was considered by Esteban and Ray (2011) in a behavioural model of conflict that provides a link between conflict, inequality and polarization (see also Chakravarty, 2015). On the other hand, as Skaperdas (1996) established, the logit function, that is, $p^i(\boldsymbol{y}) = \dfrac{e^{\theta y_i}}{\sum_{j \in N} e^{\theta y_j}}$ is the only continuous CSF that satisfies (A1)–(A5) and (A7), where $\theta > 0$ is a constant. This Hirschleifer (1989) CSF has no Nash equilibrium in pure strategies. (A systematic comparison of the properties of these two functional forms is available in Hirschleifer(1989).) Given axiom (A1), it is easy to verify that the only CSF that satisfies (A6) and (A7) is the constant function $p^i(y) = \frac{1}{n}$. But constancy of a CSF is ruled out by the assumption that $p^i(\boldsymbol{y})$ is strictly increasing in y_i and is strictly decreasing in y_j for all $j \neq i$.

The distribution-based justification of the ratio form CSF was provided by Hirshleifer and Riley (1992) and Jia (2008). The former authors suggested a derivation of the Tullock CSF for the case of two contestants under certain stochastic assumptions. Jia (2012) showed that the ratio form CSF emerges under some alternative stochastic assumptions with an arbitrary number of contestants.

Adoption of either (A6) or (A7) reflects a particular notion of value judgment. Investors may not be unanimous in their choice between these two invariance notions. If

we replace p^i by an inequality index and $\boldsymbol{y}$ by the income distribution in an n-person society, then these two invariance concepts are referred to as rightist and leftist notions of inequality invariance (Kolm, 1976). In fact, experimental questionnaire studies provide ample evidence for a middle position between these two views (Amiel and Cowell, 1992).

In the current context, the following invariance postulate, which we refer to as μ-independence axiom, represents a diversity of views concerning invariance of CSFs:

(A8) $$p^i(\boldsymbol{y} + c(\mu\, \boldsymbol{y} + (1-\mu)1^n)) = p^i(\boldsymbol{y}),$$

where $\boldsymbol{y} \in [0, \infty)^n$ is arbitrary, μ, $0 \leq \mu \leq 1$, is a parameter which reflects a contestant's view on winning probability equivalence, c is a scalar such that $\boldsymbol{y} + c(\mu\, \boldsymbol{y} + (1-\mu)1^n)$ and 1^n, the n-coordinated vector of ones, are expressed in the unit of measurement of efforts, so that $\boldsymbol{y} + c(\mu\, \boldsymbol{y} + (1-\mu)1^n)$ becomes well defined. The scale and translation invariance criteria given by (A6) and (A7) emerge as polar cases of the μ-independence notion (A8) when μ takes on the values 1 and 0 respectively. As the value of μ increases (respectively, decreases) to one (respectively, zero) the contestant becomes more concerned about scale (respectively, translation) invariance[3].

In (A6) for $\lambda > 1$, an equi-proportionate increase in investments increases their absolute gaps, whereas in (A7) for any allowable $c < 0$, an equal absolute reduction in investments increases ratios among them. In each of the situations, the probabilities of success remain the same. However, investors may not like these extreme positions and often may like to choose an intermediate position between them, which is represented by a value of $\mu \in (0, 1)$. Our axiom (A8) reflects this. As we have noted, the particular cases $\mu = 1$ and $\mu = 0$ correspond respectively to (A6) and (A7). In these polar cases the values of μ are common knowledge among the contestants. In view of this and continuity of the CSF for $\mu \in (0, 1)$ (see Theorem 1 below), we assume that the value of $\mu \in (0, 1)$ is also common knowledge.

Axiom (A8) has some similarity with Pfingsten's (1991) non-homogenous axiom employed in the context of characterization of surplus sharing. In a surplus sharing problem some agents invest in a project and on successful completion of the project the agents need to distribute the surplus among them. According to Pfingsten, Moulin's (1987) interpretation of the homogeneity condition captures only the situation when investments are measured in nominal terms. But when investments are measured in real terms, this condition becomes a value judgment and alternative views are possible. He also developed characterizations of non-homogenous parametric sharing methods which contain the homogenous and translation invariant methods as polar cases. As Pfingsten (1991) argued, we thus have a conflict between two objectives, one represents more variety in parametric value judgment and the others reflect just two extreme views and some people may not be happy with the latter. (See also Güth(1988) for similar opinion in situations of behavioral aspects of distributive justice.)

[3] In the context of income inequality measurement this axiom is the Bossert-Pfingsten (1990) μ-independence inequality axiom. For further discussion, See Seidl and Pfingsten (1997) and Del Rio and Ruiz Castillo (2000).A more recent discussion is available in Chakravarty (2015).

The following theorem isolates the CSF that satisfies (A8). We first identify the CSF for the parametric range $0 < \mu < 1$. The two extreme cases will be discussed later. We make the following assumption at the outset.

Assumption (A): *In (A5′), we assume that $f(0) > 0$ and f is continuously differentiable on $[0, \infty)$ with $f'(0) > 0$.*

In order to state the theorem formally, we now present the following definition.

Definition 1: For each $\mu \in [0, 1]$ and $\eta > 0$, let p_μ be a CSF such that

$$p_\mu^i(\mathbf{y}) = \frac{[1 + \mu(y_i - 1)]^{\frac{\eta}{\mu}}}{\sum_{j \in N} [1 + \mu(y_j - 1)]^{\frac{\eta}{\mu}}} \tag{2}$$

for all $y \in [0, \infty)^n$ and $i \in N$.

Theorem 1: *Assume that the CSF meets assumption (A) and fix $\mu \in [0, 1]$ .Then it satisfies axiom (A 5′) and μ-independence if and only if it is p_μ*

Proof: To demonstrate the necessity part, let us consider $(y_1, y_2) \in (0, \infty)^2$ and note that $p^i(\mathbf{y}) = \dfrac{f(y_i)}{f(y_1) + f(y_2)}$, where $i = 1,\ 2$. Then by (A8) we get,

$$\frac{f\left[c(1+\mu)y_1 + c(1-\mu)\right]}{f(y_1)} = \frac{f\left[c(1+\mu)y_2 + c(1-\mu)\right]}{f(y_2)}, \tag{3}$$

where for simplicity it is assumed that $c > 0$. From (3) it follows that $\dfrac{f[c(1+\mu)z + c(1-\mu)]}{f(z)}$ is independent of the effort level z. Differentiating $\dfrac{f[c(1+\mu)z + c(1-\mu)]}{f(z)}$ with respect to z we get,

$$\frac{d}{dz}\left(\frac{f\left[c(1+\mu)z + c(1-\mu)\right]}{f(z)}\right) = 0, \tag{4}$$

which implies that

$$(c\mu + 1) f(z) f'\{(c\mu + 1)z + c(1-\mu)\} = f'(z) f\{(c\mu + 1)z + c(1-\mu)\}, \tag{5}$$

where f' stands for the derivative of f.

Equation (5) holds for all finite $z > 0$. Letting $z \to 0$ on each side of (5) and applying continuity of f' we get

$$(c\mu + 1) f(0) f'\{c(1-\mu)\} = f'(0) f\{c(1-\mu)\}, \tag{6}$$

from which it follows that

$$\frac{f'\{c(1-\mu)\}}{f\{c(1-\mu)\}} = \frac{\eta}{(c\mu + 1)}, \tag{7}$$

where $\eta = \dfrac{f'(0)}{f(0)} > 0$ (since $f(0) > 0$ and $f'(0) > 0$, by assumption (A)). Integrating both sides of (7) we get,

$$\log f\{c(1-\mu)\} = \frac{\eta}{\mu}\log(c\mu+1) + k\,, \tag{8}$$

where k is the constant of integration.

From (8) it follows that

$$f\{c(1-\mu)\} = (c\mu+1)^{\frac{\eta}{\mu}} e^{k}\,. \tag{9}$$

This holds for all $c > 0$ and for all $\mu \epsilon (0, 1)$. Thus,

$$f(z) = \xi\{\mu(z-1)+1\}^{\frac{\eta}{\mu}}\,, \tag{10}$$

where $\xi = \dfrac{e^k}{(1-\mu)^{\eta/\mu}}$, $\eta > 0$ are constants. By continuity of f, the solution extends to the case where $z = 0$. Substituting this form of f into $p^i(\boldsymbol{y}) = \frac{f(y_i)}{\sum_{j\in N} f(y_j)}$ we get the desired form of the CSF. This establishes the necessity part of the theorem. The sufficiency is easy to verify. Δ

As $\mu \to 0$, $p^i_\mu(\boldsymbol{y})$ in (2) approaches $\dfrac{e^{\eta y_i}}{\sum_{j\in N} e^{\eta y_j}}$, the Hirshleifer CSF associated with (A7) (given that $\theta = \eta$). (Here for evaluating the limit we use the fact that $\lim_{z\to 0+}(1+z)^{\frac{1}{z}} = e$.) On other hand, as $\mu \to 1$, the limiting value of $p^i_\mu(\boldsymbol{y})$ given by (2) coincides with the Tullock (1980) CSF corresponding to (A6) (given that $\eta = \delta$). Thus, $p^i_\mu(\boldsymbol{y})$ in (2) may be regarded as a generalization of scale and translation invariant CSFs.

Note that the scale invariance condition (A6) can very well be relaxed to the following more general ranking property.

(A9) Scale Consistency: For all $\boldsymbol{x}, \boldsymbol{y} \in [0,\infty)^n$ if for some $i \in N$, $p^i(\boldsymbol{y}) \geq p^i(\boldsymbol{x})$ holds, then $p^i(\lambda\boldsymbol{y}) \geq p^i(\lambda\boldsymbol{x})$ for all $\lambda > 0$.

Evidently, scale invariance implies scale consistency but the converse is not true. For example, consider the CSF $p^i(\boldsymbol{y}) = \dfrac{2^{y_i}}{2^{y_1}+2^{y_2}}$ for $y = (y_1, y_2) \in [0,\infty)^2$. Then $p^i(\lambda\boldsymbol{y}) \neq p^i(y)$ for any $\lambda > 0$, $\lambda \neq 1$. However, $p^1(\boldsymbol{y}) \geq p^1(\boldsymbol{x})$ implies: $2^{y_1-y_2} \geq 2^{x_1-x_2}$, which gives, $2^{\lambda(y_1-y_2)} \geq 2^{\lambda(x_1-x_2)}$, that is, $p^1(\lambda\boldsymbol{y}) \geq p^1(\lambda\boldsymbol{x})$ for any $\lambda > 0$. Thus, if we restrict ourselves to the dimension $n = 2$, then $p^i(\boldsymbol{y})$ is scale consistent, but not scale invariant.

To cite an example in the case $n = 3$, consider the CSF $p^1(\boldsymbol{y}) = \dfrac{2^{y_1}}{2^{y_1}+2^{y_2}}$ and $p^2(\boldsymbol{y}) = \dfrac{2^{y_2-1}}{2^{y_1}+2^{y_2}} = p^3(\boldsymbol{y})$ for $\boldsymbol{y} = (y_1, y_2, y_3) \in [0,\infty)^3$. Then $p^i(\lambda\boldsymbol{y}) \neq p^i(\boldsymbol{y})$ for any $\lambda > 0, \lambda \neq 1$. However, $p^i(\boldsymbol{y}) \geq p^i(\boldsymbol{x})$ implies: $2^{y_1-y_2} \geq 2^{x_1-x_2}$, which gives, $2^{\lambda(y_1-y_2)} \geq 2^{\lambda(x_1-x_2)}$, that is, $p^i(\lambda\boldsymbol{y}) \geq p^i(\lambda\boldsymbol{x})$ for any $\lambda > 0$.

Note that satisfaction of $p^i(\lambda \boldsymbol{y}) \geq p^i(\lambda \boldsymbol{x})$ for all $\lambda > 0$ implies fulfilment of $p^i(\boldsymbol{y}) \geq p^i(\boldsymbol{x})$. Note also that if $p^i(\boldsymbol{y}) > p^i(\boldsymbol{x})$ holds, then there is at least one contestant $j \neq i$ such that $p^j(\boldsymbol{y}) < p^j(\boldsymbol{x})$ holds. The reason for this is that $\sum_{i=1}^{n} p^j(\boldsymbol{y}) = \sum_{j=1}^{n} p^j(\boldsymbol{x}) = 1$. (A9) is a ranking property in the sense that the inequality remains invariant under any increasing transformation Ω of p^i s. Furthermore, $\Omega\left(p^i\right)$ values given by $\Omega\left(p^i(\boldsymbol{y})\right) = \dfrac{\Omega\left(p^i(\boldsymbol{y})\right)}{\sum_{j \in N} \Omega\left(p^j(\boldsymbol{y})\right)}$, $i \in N$, are probabilities[4].

The next theorem demonstrates that the CSF of the power function type is the only one that fulfils (A9).For this characterization, we omit the origin from the domain of the CSF.

Theorem 2: *Assume that the number of contestants is greater than 2 and the function f is continuously differentiable on* $(0, \infty)$. *Then, given assumption* (A), *the CSF satisfies axioms* (A1)–(A5) *and* (A9) *if and only if it is of the Tullock* (1980) *form given by*

$$p^i(\boldsymbol{y}) = \frac{y_i^{\delta}}{\sum_{j \in N} y_j^{\delta}}, \tag{11}$$

where $\delta > 0$ is a constant, $y \neq 01^n \in (0, \infty)^n$ and $i \in N$ are arbitrary.

Proof: By Theorem 1 of Skaperdas (1996), axioms (A1)–(A5) are satisfied if and only if the CSF is given by (A 5′). Observe that for any $\boldsymbol{y} = (y_1, y_2) \in (0,\infty)^2$ we have, $p^1(\boldsymbol{y}) = \dfrac{f(y_i)}{f(y_1) + f(y_2)}$, where $i = 1, 2$. Consider $\left(y_1', y_2'\right), (\bar{y}_1, \bar{y}_2) \in (0, \infty)^2$. Then $p^1\left(\boldsymbol{y}'\right) \geq p^1(\bar{y})$ is same as $\dfrac{f\left(y_1'\right)}{f\left(y_1'\right) + f\left(y_2'\right)} \geq \dfrac{f(\bar{y}_1)}{f(\bar{y}_1) + f(\bar{y}_2)}$, that is, if and only if $\dfrac{f\left(y_2'\right)}{f\left(y_1'\right)} \leq \dfrac{f(\bar{y}_2)}{f(\bar{y}_1)}$. Thus, by (A9) we have,

$$\frac{f\left(y_2'\right)}{f\left(y_1'\right)} \leq \frac{f\left(\bar{\boldsymbol{y}}_2\right)}{f\left(\bar{\boldsymbol{y}}_1\right)} \text{ if and only if } \frac{f\left(\lambda y_2'\right)}{f\left(\lambda y_1'\right)} \leq \frac{f\left(\lambda \bar{\boldsymbol{y}}_2\right)}{f\left(\lambda \bar{\boldsymbol{y}}_1\right)} \text{ for all } \lambda > 0\,. \tag{12}$$

Now, we claim that $\dfrac{f(\lambda y_2)}{f(\lambda y_1)} = F_\lambda\left(\dfrac{f(y_2)}{f(y_1)}\right)$ for some non-decreasing function F_λ. To demonstrate this, consider, as before, two distinct effort vectors $\left(y_1', y_2'\right), (\bar{y}_1, \bar{y}_2) \in (0, \infty)^2$. Then we have,

$$\frac{f\left(\lambda \bar{\boldsymbol{y}}_2\right)}{f\left(\lambda \bar{\boldsymbol{y}}_1\right)} = \frac{f\left(\lambda y_2'\right)}{f\left(\lambda y_1'\right)} \text{ if and only if } \frac{f\left(\bar{\boldsymbol{y}}_2\right)}{f\left(\bar{\boldsymbol{y}}_1\right)} = \frac{f\left(y_2'\right)}{f\left(y_1'\right)}\,.$$

[4] (A9) becomes Zheng's (2007) unit consistency axiom if we replace p^i by an inequality index, $\boldsymbol{y}$ and $\boldsymbol{x}$ by income distributions in two $n-$person societies and the weak inequality $\geq$ by the strict inequality $>$ (see also Chakravarty 2015).

This implies that $\dfrac{f(\lambda y_2)}{f(\lambda y_1)}$ is a function of $\dfrac{f(y_2)}{f(y_1)}$. Non-decreasingness of this function is a consequence of (12).

Define

$$u_\lambda(y_1, y_2) = \frac{f(\lambda y_2)}{f(\lambda y_1)} \tag{13}$$

and

$$q(y_1, y_2) = \frac{f(y_2)}{f(y_1)}. \tag{14}$$

Note that since f is strictly increasing, it is invertible so that by the definition of F_λ we have,

$$F_\lambda\left(\frac{f(y)}{f(1)}\right) = \frac{f(\lambda y)}{f(\lambda)}, \tag{15}$$

which in turn implies that

$$F_\lambda(t) = \frac{f\left(\lambda f^{-1}(tf(1))\right)}{f(\lambda)} \tag{16}$$

for all $t > 0$.

As f is differentiable, it follows that so is F_λ.

Now, we claim that the Jacobian of u_λ and q with respect to y_1 and y_2 must vanish. To establish this, simply observe that $u_\lambda = F_\lambda . q$ so that

$$\begin{vmatrix} \dfrac{\partial u_\lambda}{\partial y_1} & \dfrac{\partial u_\lambda}{\partial y_2} \\ \dfrac{\partial q}{\partial y_1} & \dfrac{\partial q}{\partial y_2} \end{vmatrix} = \begin{vmatrix} \dfrac{dF_\lambda}{dq}.\dfrac{\partial q}{\partial y_1} & \dfrac{dF_\lambda}{dq}\dfrac{\partial q}{\partial y_2} \\ \dfrac{\partial q}{\partial y_1} & \dfrac{\partial q}{\partial y_2} \end{vmatrix} = \frac{dF_\lambda}{dq}\begin{vmatrix} \dfrac{\partial q}{\partial y_1} & \dfrac{\partial q}{\partial y_2} \\ \dfrac{\partial q}{\partial y_1} & \dfrac{\partial q}{\partial y_2} \end{vmatrix} = 0. \tag{17}$$

This implies that

$$\frac{f(\lambda y_2) f'(\lambda y_1) f'(y_2)}{f(\lambda y_1)} = \frac{f'(y_1) f(y_2) f'(\lambda y_2)}{f(y_1)}. \tag{18}$$

Equation (18) can be rearranged as

$$\frac{f'(\lambda y_1)}{f(\lambda y_1)}.\frac{f'(y_2)}{f(y_2)} = \frac{f'(\lambda y_2)}{f(\lambda y_2)}.\frac{f'(y_1)}{f(y_1)}. \tag{19}$$

Now, (19) holds for all $(y_1, y_2) \in (0, \infty)^2$. Putting $y_1 = z > 0$, $y_2 = 1$ in (19) and letting $h(z) = \dfrac{f'(z)}{f(z)}$ we get,

$$h(\lambda z) h(1) = h(z) h(\lambda). \tag{20}$$

Given that f is positive valued on $(0, \infty)$ and increasing, h is positive. It is continuous as well. Since (20) holds for all positive z and λ, it is a fundamental Cauchy equation, of which the only continuous solution is given by

$$h(z) = K_1 z^{\alpha} \tag{21}$$

for some $K_1 > 0$ and α is a real number (Aczel, 1966, p. 41, Theorem 3).

Case I: $\alpha \neq -1$
Then (21) yields:

$$\frac{f'(z)}{f(z)} = K_1 z^{\alpha} . \tag{22}$$

Integrating both sides of (22) we get,

$$\log(f(z)) = K z^{\alpha+1} + K' , \tag{23}$$

where $K = \dfrac{K_1}{\alpha+1}$ and K' is the constant of integration.

Equation (23) is equivalent to

$$f(z) = A B^{z^{\beta}} , \tag{24}$$

where $A = e^{K'} > 0$, $B = e^{K} > 0$ and $\beta = 1 + \alpha$ is a non-zero real number.

Case II: $\alpha = -1$.
Then (22) becomes:

$$\frac{f'(z)}{f(z)} = K_1 z^{-1} , \tag{25}$$

which, on integration, gives

$$\log(f(z)) = K_1 \log(z) + K' , \tag{26}$$

where K' is the constant of integration.

This gives

$$f(z) = A z^{B} , \tag{27}$$

where $A = e^{K'} > 0$ and B is a real number. Since f is strictly increasing, we further require the restriction $B > 0$.

Plugging the forms of f given by (24) and (27) into $p^i(\boldsymbol{y}) = \dfrac{f(y_i)}{\sum_{j \in N} f(y_j)}$, we get the following forms of $p^i(\boldsymbol{y})$:

$$p^i(\boldsymbol{y}) = \begin{cases} \dfrac{B^{y_i^{\beta}}}{\sum_{j \in N} B^{y_j^{\beta}}} , \\ \dfrac{y_i^{B}}{\sum_{j \in N} y_j^{B}} . \end{cases} \tag{28}$$

Out of these two solutions, only the latter satisfies (A9) if $n > 2$. To see a counter

example where $n = 3$, for the first functional form in (28), take $B = 2$ and $\beta = 1$. Let $y = \left(1, \frac{\log 14}{\log 2}, \frac{\log 6}{\log 2}\right)$ and $x = \left(1, 4, \frac{\log 3}{\log 2}\right)$. Note that $p^1(\boldsymbol{y}) = \frac{2}{2+14+6} = \frac{1}{11}$ and $p^1(\boldsymbol{x}) = \frac{2}{2+16+3} = \frac{2}{21}$ so that $p^1(\boldsymbol{y}) < p^1(\boldsymbol{x})$. But $p^1(2\boldsymbol{y}) = \frac{2^2}{2^2+14^2+6^2} = \frac{4}{236}$ and $p^1(2\boldsymbol{x}) = \frac{2^2}{2^2+16^2+3^2} = \frac{4}{269}$ implying that $p^1(2\boldsymbol{y}) > p^1(2\boldsymbol{x})$. Thus the CSF fails to satisfy (A9).

Putting $B = \delta$ in the second functional form in (28), we get the Tullock form of CSF given by (11). This completes the necessity part of the proof of the theorem. The sufficiency can be easily verified by checking that the CSF given by (17) fulfils (A1)–(A5) and (A9). Δ

Combining Theorem 2 of Skaperdas (1996) and Theorem 2 of this paper we arrive at the following result:

Theorem 3: *Assume that the number of contestants is greater than 2 and assumption (A) holds. Then the following statements are equivalent*:

(i) The CSF satisfies axioms (A1)–(A6).
(ii) The CSF satisfies axioms (A1)–(A5) and (A9).
(iii) The CSF is of the Tullock form given by (11).

Remark 1: Hillman and Riley (1989) considered a model of political process with uncertain impacts of efforts. Given that the number of contestants is two, they derived a CSF for which the effective investment function is of the form y_i^{δ}. Micro-foundations for a subset of the underlying CSFs for innovative tournaments and patent races were offered by Fullerton and McAfee (1999) and Baye and Hoppe (2003). Since we assume at the outset that $n \geq 3$, there is an important difference between the Hillman-Riley framework and that considered in this paper.

We next consider the following ranking counterpart to (A7):

(A10) Translation Consistency: For all $\boldsymbol{x}, \boldsymbol{y} \in [0, \infty)^n$ if, if for some $i \in N$, $p^i(\boldsymbol{y}) \geq p^i(\boldsymbol{x})$ holds, then $p^i(\boldsymbol{y} + c1^n) \geq p^i(\boldsymbol{x} + c1^n)$, where 1^n is the n-coordinated vector of ones and c is a scalar such that $y_i + c \geq 0$ for all $i \in N$.

Evidently, (A7) is sufficient but not necessary for (A10). Like (A9), (A10) is also a ranking property.

Remark 2: Fix $\boldsymbol{x} \in [0, \infty)^n$ and define $Y_x^* = \{\boldsymbol{y} \in R^n$: there is $i \in N$ such that $y_i \geq x_i$ and $y_j \leq x_j$ for all $j \neq i\}$ Then for all $\boldsymbol{y} \in Y_x^*$ we have, $p^i(\boldsymbol{y}) \geq p^i(\boldsymbol{x})$. Also, $\boldsymbol{y} \in Y_x^*$ implies: $\lambda\boldsymbol{y} \in Y_x^*$ for all $\lambda > 0$ and $\boldsymbol{y} + c1^n \in Y_x^*$ for all $c > 0$. From this it follows that $p^i(\lambda\boldsymbol{y}) \geq p^i(\lambda\boldsymbol{x})$ and $p^i(\boldsymbol{y} + c1^n) \geq p^i(\boldsymbol{x} + c1^n)$. This observation, however, implies neither axiom (A9) nor (A10). This is because $p^i(\lambda\boldsymbol{y}) \geq p^i(\lambda\boldsymbol{x})$ never implies that $\boldsymbol{y} \in Y_x^*$.

In the following theorem we characterize the entire class of CSFs that are translation consistent.

Theorem 4: *Assume that the number of contestants is greater than 2 and the function f meets assumption* (A) *Then the CSF satisfies axioms (A 5′) and* (A10) *if and only if it is of the Hirschleifer* (1989) *form given by*:

$$p^i(\boldsymbol{y}) = \frac{e^{\theta y_i}}{\sum_{j \in N} e^{\theta y_j}} \tag{29}$$

where θ is a positive constant, $\boldsymbol{y} \in (0, \infty)^n$ and $i \in N$ are arbitrary.

Proof: Take, as in the proof Theorem 2, $\left(y_1', y_2'\right), (\bar{y}_1, \bar{y}_2) \in (0, \infty)^2$. Then $p^1(\boldsymbol{y}') \geq p^1(\bar{\boldsymbol{y}})$ is same as, $\frac{f(y_2')}{f(y_1')} \leq \frac{f(\bar{y}_2)}{f(\bar{y}_1)}$.

By (A10),

$$\frac{f(y_2')}{f(y_1')} \leq \frac{f(\bar{y}_2)}{f(\bar{y}_1)} \text{ if and only if } \frac{f(y_2'+c)}{f\left(y_1'+c\right)} \leq \frac{f(\bar{y}_2+c)}{f(\bar{y}_1+c)} \text{ for all } c > 0\,. \tag{30}$$

As in the proof of Theorem 2, one can easily see that there exists a continuous and non-decreasing function G_c such that

$$\frac{f(y_2+c)}{f(y_1+c)} = G_c\left(\frac{f(y_2)}{f(y_1)}\right). \tag{31}$$

Define

$$w_c(y_1, y_2) = \frac{f(y_2+c)}{f(y_1+c)}. \tag{32}$$

Since w_c and q are functionally related, the Jacobian of w_c and q in (14) with respect to y_1 and y_2 must vanish. This implies that

$$\frac{f'(y_1+c)}{f(y_1+c)} \cdot \frac{f'(y_2)}{f(y_2)} = \frac{f'(y_2+c)}{f(y_2+c)} \cdot \frac{f'(y_1)}{f(y_1)}. \tag{33}$$

Equation (33) holds for all $(y_1, y_2) \in (0, \infty)^2$. Putting $y_1 = z > 0$, $y_2 = \varepsilon > 0$ and substituting $\frac{f'(z)}{f(z)}$ by $\psi(z)$, which is positive on $(0, \infty)$, we get

$$\psi(z+c)\,\psi(\varepsilon) = \psi(z)\,\psi(c+\varepsilon)\,. \tag{34}$$

Letting $\varepsilon \to 0$ in (34) and using continuous differentiability of f we get,

$$\psi(z+c)\,\psi(0) = \psi(z)\,\psi(c)\,. \tag{35}$$

From (35) it follows that$\psi(0) > 0$. This equation holds for all positive z and c. The only continuous solution to (35) is given by

$$\psi(z) = \upsilon e^{\rho} z \tag{36}$$

for some positive $\upsilon (= \psi(0))$ and real ρ (see Aczel, 1966, p. 84). By continuity of ψ, the solution extends to the case when $z = 0$.

From (36) it is evident that

$$\frac{f'(z)}{f(z)} = \upsilon e^{\rho z}. \tag{37}$$

Case I: $\rho \neq 0$.
Integrating both sides of (37) we get,

$$\log(f(z)) = K_3 e^{\rho z} + K_4, \tag{38}$$

where $K_3 = \dfrac{\nu}{\rho}$ and K_4 is the constant of integration.

From (38) it immediately follows that

$$f(z) = E H^{e^{\rho z}}, \tag{39}$$

where $E = e^{K_4}$ and $H = e^{K_3 \upsilon}$ are positive constants.

Case II: $\rho = 0$.
Then (37) becomes:

$$\frac{f'(z)}{f(z)} = \upsilon. \tag{40}$$

Integrating both sides of (40) we get,

$$\log f(z) = \upsilon z + C \tag{41}$$

where C is the constant of integration.

Equation (41) is equivalent to:

$$f(z) = Q\, e^{\upsilon z}, \tag{42}$$

where $Q = e^C > 0$. For strict increasingness of f we need the restriction $\upsilon > 0$. Substituting the forms of f given by (39) and (42) in $p^i(\boldsymbol{y}) = \dfrac{f(y_i)}{\sum_{j \in N} f(y_j)}$, the resulting forms of $p^i(y)$ become:

$$p^i(\boldsymbol{y}) = \begin{cases} \dfrac{H^{e^{\rho y_i}}}{\sum_{j \in N} H^{e^{\rho y_j}}}, \\ \dfrac{e^{\upsilon y_i}}{\sum_{j \in N} e^{\upsilon y_j}}. \end{cases} \tag{43}$$

However, it can be easily checked that the former violates (A10) if $n > 2$. To see a counter example for dimension $n = 3$, consider the CSF given by the first functional form in (43). Take $H = 2$, $\rho = 1$ and $c = \log 2$. Let $\boldsymbol{y} = \left(0, \dfrac{\log 14}{\log 2}, \log\left(\dfrac{\log 6}{\log 2}\right)\right)$ and $x = \left(0, \log 4, \log\left(\dfrac{\log 6}{\log 2}\right)\right)$. Note that $p^1(\boldsymbol{y}) = \dfrac{2}{2+14+6} = \dfrac{1}{11}$ and $p^1(\boldsymbol{x}) =$

$\frac{2}{2+16+3} = \frac{2}{21}$ so that $p^1(\boldsymbol{y}) < p^1(\boldsymbol{x})$. But $p^1(\boldsymbol{y} + \boldsymbol{c}1^3) = \frac{2^2}{2^2+14^2+6^2} = \frac{4}{236}$ and $p^1(\boldsymbol{x} + c1^3) = \frac{2^2}{2^2+16^2+3^2} = \frac{4}{269}$ implying that $p^1(\boldsymbol{y} + c1^3) > p^1(\boldsymbol{x} + c1^3)$. Thus the CSF fails to satisfy (A10).

Putting $\upsilon = \theta$ in the second functional form specified in (43) we arrive at the CSF given by (29). Hence the necessity part of the theorem is demonstrated. The sufficiency follows easily. Δ

Remark 3: However, it is easy to check that for $n = 2$, both forms of CSFs mentioned in (43) satisfy (A10). Thus, in this case we get a CSF distinct from the Hirschleifer form.

Theorem 3 of Skaperdas (1996) and Theorem 4 of this paper can now be combined to yield the following result:

Theorem 5: *Assume that the number of contestants is greater than* 2 *and assumption* (A) *holds. Then the following statements are equivalent*:

(i) The CSF satisfies axioms (A1)–(A5) and (A7).
(ii) The CSF satisfies axioms (A1)–(A5) and (A10).
(iii) The CSF is of the Hirshleifer form given by (29).

Instead of considering scale consistency (A9) or Translation consistency (A10), we can also consider the following intermediate form of consistency, which is clearly a ranking counterpart of (A9).

(A11) **Intermediate μ-Consistency:** For all $\boldsymbol{x}, \boldsymbol{y} \in [0, \infty)^n$ if, for some $i \in N$, $p^i(\boldsymbol{y}) \geq p^i(\boldsymbol{x})$ holds, then

$$p^i\left(\boldsymbol{y} + c\left(\mu\, \boldsymbol{y} + (1-\mu)1^n\right)\right) \geq p^i\left(\boldsymbol{x} + c\left(\mu\, \boldsymbol{y} + (1-\mu)1^n\right)\right)$$

where $\mu \in [0, 1]$ is a parameter and $c \in R$ is a scalar such that

$$\boldsymbol{x} + c\left(\mu\, \boldsymbol{y} + (1-\mu)1^n\right), \quad \boldsymbol{y} + c\left(\mu\, \boldsymbol{y} + (1-\mu)1^n\right) \in [0, \infty)^n .$$

We now characterize all CSFs satisfying intermediate μ-consistency.

Theorem 6: *Assume that the number of contestants is greater than* 2 *and let the function f meet assumption* (A). *Then the* CSF *satisfies axioms* (A5′) *and* (A11) *if and only if it is of the form p_μ.*

Proof: Take, as in the proofs of theorems 2 and 4, $(y_1', y_2'), (\bar{y}_1, \bar{y}_2) \in [0, \infty)^2$. By (A11) it follows that

$$\frac{f(y_2')}{f(y_1')} \leq \frac{f(\bar{y}_2)}{f(\bar{y}_1)} \textit{ iff } \frac{f\left\{y_2' + c\,(\mu\, y_2' + 1 - \mu)\right\}}{f\left\{y_1' + c\,(\mu\, y_1' + 1 - \mu)\right\}} \leq \frac{f\left\{\bar{y}_2 + c\,(\mu\, \bar{y}_2 + 1 - \mu)\right\}}{f\left\{\bar{y}_1 + c\,(\mu\, \bar{y}_1 + 1 - \mu)\right\}} \tag{44}$$

for all $c > 0$.

Therefore, for all $y_1,\ y_2 \in (0, \infty)$ we have,

$$\frac{f\left\{y_2 + c\,(\mu\, y_2 + 1 - \mu)\right\}}{f\left\{y_1 + c\,(\mu\, y_1 + 1 - \mu)\right\}} = H_\mu\left(\frac{f(y_2)}{f(y_1)}\right) \tag{45}$$

for some continuous and non-decreasing function H_μ.

Define

$$v_c(y_1,\ y_2) = \frac{f\,\{y_2 + c\,(\mu\, y_2 + 1 - \mu)\}}{f\,\{y_1 + c\,(\mu\, y_1 + 1 - \mu)\}} \tag{46}$$

and

$$r(y_1,\ y_2) = \frac{f(y_2)}{f(y_1)}. \tag{47}$$

Since v_c and r are functionally related, the Jacobian of v_c and r with respect to y_1 and y_2 must vanish. That is,

$$\begin{vmatrix} \dfrac{\partial v_c}{\partial y_1} & \dfrac{\partial v_c}{\partial y_2} \\ \dfrac{\partial r}{\partial y_1} & \dfrac{\partial r}{\partial y_2} \end{vmatrix} = 0 \tag{48}$$

Simplifying and rearranging we get,

$$\frac{f'\,\{y_2 + c\,(\mu y_2 + 1 - \mu)\}}{f\,\{y_2 + c\,(\mu y_2 + 1 - \mu)\}}\,\frac{f'(y_1)}{f(y_1)} = \frac{f'\,\{y_1 + c\,(\mu y_1 + 1 - \mu)\}}{f\,\{y_1 + c\,(\mu y_1 + 1 - \mu)\}}\,\frac{f'(y_2)}{f(y_2)}. \tag{49}$$

For $z \in (0, \infty)$, put

$$h(z) = \frac{f'(z)}{f(z)}\,. \tag{50}$$

Then h is positive-valued (since f is positive and strictly increasing).

From (49) it follows that

$$\frac{h\ \{y_2 + c\,(\mu y_2 + 1 - \mu)\}}{h\ \{y_1 + c\,(\mu y_1 + 1 - \mu)\}} = \frac{h\,(y_2)}{h\,(y_1)}\,. \tag{51}$$

This holds for all $y_1,\ y_2 \in (0, \infty)$. Putting $y_2 = z$ and $y_1 = 0$ we get,

$$\frac{h\ ((1 + c\mu)\ z + c(1 - \mu))}{h\,(c(1 - \mu))} = \frac{h(z)}{h(0)}\,. \tag{52}$$

Put

$$\ln\left(\frac{h(z)}{h(0)}\right) = \phi(z)\,, \tag{53}$$

which gives

$$\phi(0) = 0\,. \tag{54}$$

Then (52) yields:

$$\phi\ \{(1 + c\mu)\,z + c\,(1 - \mu)\} = \phi\,(z) + \phi\ (c(1 - \mu)) \tag{55}$$

which implies that

$$\frac{\phi\{(1-c\mu)z+c(1-\mu)\}-\phi\{c(1-\mu)\}}{(1+c\mu)z}=\frac{\phi(z)-\phi(0)}{(1+c\mu)z}. \tag{56}$$

Proceeding to limits of both sides as $z \to 0$ we have,

$$\phi'\{c(1-\mu)\} = \frac{1}{(1+c\mu)}\phi'(0). \tag{57}$$

Differentiating both sides of (55) we get,

$$\phi'(z) = \frac{h'(z)}{h(z)}. \tag{58}$$

Substituting ϕ' from (58) into (57) we get,

$$\frac{h'\{c(1-\mu)\}}{h\{c(1-\mu)\}} = \frac{\eta_0}{1+c\mu}, \tag{59}$$

where $\eta_0 = \dfrac{h'(0)}{h(0)}$.

Case I: $h'(0) = 0$.

Then $\eta_0 = 0$ and from (59) it follows that $h'(t) = 0$ for all $t \in [0, \infty)$. Consequently, $h(t) = c_1$ for some positive constant c_1. This, in turn implies that

$$\frac{f'(t)}{f(t)} = c_1 \tag{60}$$

for all $t \in [0, \infty)$. Integrating both sides of (60) we get,

$$f(t) = c_2 e^{c_1 t}, \tag{61}$$

where $c_2 > 0$ is a constant.

Case II: $h'(0) \neq 0$.

Proceeding as in the proof of Theorem 1 we can show that

$$h(z) = \kappa\{\mu(z-1)+1\}^{\frac{\eta_0}{\mu}} \tag{62}$$

for some constant $\kappa > 0$.

Using (59) we have,

$$\frac{f'(z)}{f(z)} = \kappa\{\mu(z-1)+1\}^{\frac{\eta_0}{\mu}}. \tag{63}$$

If $\dfrac{\eta_0}{\mu} \neq -1$, then integrating both sides of (63) we get,

$$\log f(z) = \kappa\frac{1}{\left(\dfrac{\eta_0}{\mu}+1\right)}\{\mu(z-1)+1\}^{\frac{\eta_0}{\mu}} + \chi_1, \tag{64}$$

where χ_1 is the constant of integration.

Thus,

$$f(z) = \chi \exp\left[\kappa \frac{1}{\left(\frac{\eta_0}{\mu}+1\right)} \{\mu(z-1)+1\}^{\frac{\eta_0}{\mu}}\right], \tag{65}$$

where $\chi > 0$.

On the other hand, if $\frac{\eta_0}{\mu} = -1$, then (63), on integration, yields:

$$\log f(z) = \kappa \frac{1}{\mu} \log\{\mu(z-1)+1\} + \chi_1, \tag{66}$$

which implies that

$$f(z) = \chi \{\mu(z-1)+1\} e^{\frac{\kappa}{\mu}} \tag{67}$$

for some constant $\chi > 0$.

Now if $n > 2$, then it is easy to see that out of the forms of CSF underlying (61), (65) and (67), only (67) is in conformity with (A11). Substituting κ by η we catch hold of the CSF given by (2). This completes the proof of the necessity part of the Theorem. The sufficiency can be checked easily. Δ

We are now in a position to state the following:

Theorem 7: *Assume that the number of contestants is greater than* 2 *and assumption* (A) *holds. Then the following statements are equivalent*:

(i) *The CSF satisfies axioms* (A1)–(A5) *and* (A9).
(ii) *The CSF satisfies axioms* (A1)–(A5) *and* (A11).
(iii) *The CSF is of the functional form given by* (2).

Remark 4: This theorem shows that the CSF given by (2), which contains the Tullock CSF as a special case and the Hirshleifer CSF as a limiting case, can be characterized by two different sets of axioms, namely {A1–A5, A9} and {A1–A5, A 11}. These two different characterizations enhance our understanding of CSF p^i_μ given by (2). Further, given that the number of contestants is greater than two, the weaker versions of the axioms are enough to motivate the Tullock and Hirshleifer CSFs.

Remark 5: However, it is easy to check that in dimension $n = 2$, all the forms of p^i resulting from (61), (65) and (67) satisfy (A9). Thus, in this case there are CSFs other than the one given by (2).

3. EQUILIBRIUM ANALYSIS

Out of the CSFs characterized in the previous sections, only the one given by (2) (that is the one satisfying μ-independence) is a new one, which has not yet been explored in the literature.

To get rid of the problem of definition at the origin, arising in the context of axioms (A4) and (A5), Corchon (2007) suggested the use of the following functional form of the CSF:

$$p^i(\mathbf{y}) = \begin{cases} \dfrac{f(y_i)}{\sum\limits_{j\in N} f\left(y_j\right)} & \text{if } \sum\limits_{j\in N} f\left(y_j\right) > 0\,, \\ \dfrac{1}{n}, & \text{otherwise}\,, \end{cases} \tag{68}$$

where $y \in [0, \infty)^n$ and $i \in N$ arbitrary.

The function $f : [0, \infty) \to [0, \infty)$ is assumed to obey the following properties:

(i) f is twice continuously differentiable in $(0, \infty)$.
(ii) f is concave.
(iii) f is strictly increasing.
(iv) $f(0) = 0$ and $\lim\limits_{z\to\infty} f(z) = \infty$.
(v) $\dfrac{zf'(z)}{f(z)}$ is bounded for all $z \in (0, \infty)$.

The functional form (68) along with properties (i)–(v) will be required in the sequel in our quest for the existence of a Nash equilibrium in a particular situation. It can be easily checked that the CSF p^i_μ given by (2) satisfies all the conditions (i)–(v) mentioned above except for condition (iv), since $f(0) > 0$ (unless $\mu = 1$).

So, it will now be worthwhile to investigate whether this CSF supports a Nash equilibrium in efforts. Let $V_i = V_i(y_1, y_2, \ldots, y_n)$ be the value of the prize obtained by the ith contestant and $C_i(y_i)$ be the cost attributed by i to his action y_i As in Corchon (2007), we make the following assumptions:

a) All agents have the same cost function C, that is, $C_i = C$ for all i.
b) The common functional form of V_i is the following:
$V = V_0 + a\sum\limits_{i=1}^{n} f(y_i)$, where $V_0 > 0$, $a \geq 0$.
c) There exists $(\overline{z}, \tau)$ such that for all $z > \overline{z}$ we have, $(af'(z) - C'(z)) < \tau < 0$.

It may be noted that there are no well-founded criteria to guide the choice of a cost function here[5]. The quantity V_0 in (b) may be regarded as fixed prize.

Corchon (2007) showed that under the above assumptions there is a symmetric equilibrium. Nti (1997) obtained this result under the assumptions that $a = 0$ and $C_i(z_i) = z_i$. Szidarovskyand Okuguchi (1997) developed a generalization of this result for a CSF of the form $p_i = \dfrac{f_i(y_i)}{\sum\limits_{j=1}^{n} f_j\left(y_j\right)}$ if $\sum\limits_{j=1}^{n} f_j\left(y_j\right) > 0$ and $p_i = \frac{1}{n}$ if $\sum\limits_{j=1}^{n} f_j\left(y_j\right) = 0$.

Since our major objective in this paper is to characterize CSFs satisfying alternative

[5] This discussion does not apply to characterizations of CSFs where groups are contestants, since there is no relation between individual effort and group performance (see Münster, 2009).

invariance axioms, we will develop our analysis using the Corchon framework. Following Proposition 3.1 of Corchon (2007), we maintain that there is a Nash equilibrium if and only if the equation

$$f'(z)\left(a + V_0 \frac{n-1}{f(z)\,n^2}\right) - C'(z) = 0 \tag{69}$$

has a solution.

In the μ-independence case, by (10) we have,

$$f(0) = \xi\,(1-\mu)^{\frac{\eta}{\mu}} > 0. \tag{70}$$

In view of (70) it may appear that there is no NE since Corchon (2007) tacitly makes use of the assumption that $f(0) = 0$ in establishing the existence of NE. However, our search for NE adopts a different route, as can be seen in the sequel.

Differentiating (10) twice with respect to z we get,

$$f'(z) = \xi\eta\,\{\mu\,(z-1)+1\}^{\frac{\eta}{\mu}-1} \tag{71}$$

and

$$f''(z) = \xi\eta\,(\eta-\mu)\,\{\mu\,(z-1)+1\}^{\frac{\eta}{\mu}-2}\,. \tag{72}$$

From (71) it is immediate that $f'(z) > 0$ for all $z \in (0,\infty)$. Also, for $\eta \leq \mu$ we observe that $f''(z) < 0$ for all $z \in (0,\infty)$ so that f is concave. Finally, it is clear that

$$\lim_{z\to\infty} f(z) = +\infty. \tag{73}$$

Denoting LHS of (69) by $\psi(z)$ we see that

$$\psi(0) = f'(0)\left(a + V_0 \frac{n-1}{f(0)\,n^2}\right) - C'(0) > 0 \tag{74}$$

In view of (72) and the assumption that $((af'(z) - C'(z)) < \tau < 0$ for sufficiently large z, $\psi(z)$ becomes negative as $z \to +\infty$. So, there exists a solution to (69), which in turn implies that there is a Nash equilibrium, if $\eta \leq \mu$. Differentiation of the left hand side of (69) shows that ψ is strictly decreasing so that the solution to (69), that is, the Nash equilibrium is unique.

We summarize these findings in the following proposition, whose proof bears similarity with that of Proposition 3.1 of Corchon (2007) :

Proposition 1: Under the assumptions (a)–(c) stated above, the contest game with the CSF p_μ has a unique Nash equilibrium if $\eta \leq \mu$.

The payoff function associated with this game can be obtained by using the form of f given by (10) in V. Note that that for $y_i = 0$ we have, $p^i(y) > 0$. Proposition 1 shows that in such a situation, there is a possibility of existence of a corner solution. Clearly, this is not the case with the power function $p^i(\boldsymbol{y}) = \dfrac{y_i^\delta}{\sum_{j\in N} y_j^\delta}$, since in this case we have,

$p^i(y) = 0$ for $y_i = 0$. It is worth mentioning here that all the properties of the function f specified in Corchon (2007) are satisfied whenever $\delta \in (0, 1)$. By Proposition 1, this clearly establishes that a unique Nash equilibrium exists. However, for the logit function $p^i(\boldsymbol{y}) = \dfrac{e^{\theta y_i}}{\sum_{j \in N} e^{\theta y_j}}$ we have, $f''(z) = \theta^2 e^{\theta z} > 0$ for all $z \in (0, \infty)$. Thus, f is never concave and hence the proof of Proposition 3.1 of Corchon (2007) sheds no light on the existence of Nash equilibrium.

In view of the above discussion we can now state the following:

Remark 6: In a pure μ-independence situation, that is, when $0 < \eta \leq \mu < 1$, Nash equilibrium may emerge as a corner solution.

Cornes and Riley (2012) showed how it is possible to model a contest as a simultaneous-move game and presented an example which possesses a symmetric CSF and identical risk –averse individuals but has multiple equilibria. They also showed that symmetric contests have symmetric equilibrium and that additional conditions are necessary for unique equilibrium of a general contest. (See also Szidarovszky and Okuguchi, 1997, and Cornes and Riley, 2005.)

The existence and uniqueness of pure-strategy Nash equilibrium, and social efficiency of the equilibrium for the proportional allocation rule, a special case of the Tullock CSF, was investigated extensively with reference to communication network by Kelly (1997) and Johari and Tsitsiklis (2004). CSFs for rent seeking were studied, among others, by Corchon (2000). He proposed a model in which agents can choose between productive and rent-seeking activities. Two possible institutions considered are autocracy where taxes are fixed by the king, and parliament rule where taxes are set by majority voting. It is shown that under parliament rule there is equilibrium with no rent-seekers. However, there is another equilibrium in which the rent-seekers dominate the parliament and the tax rate is the same as that under autocracy. It also demonstrated that rent-seekers may be interested in removing autocracy. Rent-seeking activities are financed by taxing productive activities. Nonetheless, these results do not directly apply to our framework because we consider only a single homogenous group whose members only differ with respect to their investments. No product market has been directly taken into consideration and any role of voting is absent here.

4. CONCLUSION

Axiomatic characterizations of contest success functions enable us to understand them in an intuitively reasonable way in the sense that necessary and sufficient conditions are identified to isolate them uniquely. Skaperdas (1996) characterized the Tullock and Hirschleifer forms of success functions. In this paper we have substantially extended the characterizations of Skarpedas (1996) by considering a general axiom (μ-independence) and three more axioms viz. scale, translation and intermediate μ-consistencies, which involve ranking, a characteristic that has not been explored earlier in the literature. It has been shown that if the number of contestants in the game is at

least 3, the Tullock and Hirschleifer functional forms are the only functional forms satisfying respectively scale and translation consistencies. The consistency axioms which are simple and elegant may be considered as the most fundamental contributions of the paper. We also look at the possibility of existence of Nash equilibria, including the ones that may turn out as corner solutions, in different situations. It may a be worthwhile exercise to identify the class of CSFs satisfying the consistency axioms (in absence of other postulates). We leave this project as a future research program.

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