Title	MEASURING IMPROVEMENT IN WELL-BEING
Sub Title	
Author	CHAKRAVARTY, Satya R. MUKHERJEE, Diganta
Publisher	Keio Economic Society, Keio University
Publication year	1999
Jtitle	Keio economic studies Vol.36, No.2 (1999.) ,p.65- 79
JaLC DOI	
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Notes	
Genre	Journal Article
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=AA00260492-19990002-0065

KEIO ECONOMIC STUDIES **36**(2), 65–79 (1999)

MEASURING IMPROVEMENT IN WELL-BEING†

Satya R. CHAKRAVARTY and Diganta MUKHERJEE

Indian Statistical Institute, Calcutta, India

First version received November 1998; final version accepted August 1999

Abstract: A measure of improvement in well-being aggregates increments in the attainment levels of different quality-of-life attributes. This paper first characterizes the entire family of additive improvement indices, where additivity requires that the overall index can be expressed as the arithmetic average of attribute-wise indices. Then we suggest a general family of nonadditive improvement indices, of which the Tsui (1996) index becomes a particular case. Both additive and nonadditive measures are shown to have their respective purposes.

JEL Classification Numbers: D63, H53, I31, O1
Key words: Well-being, Improvement, Indices

1. INTRODUCTION

A measure of improvement in well-being with respect to some quality-of-life attribute (e.g. life expectancy, educational attainment) is a summary statistic indicating increase in the attainment or achievement level of the attribute. Such an index is often used for comparison of well-being across countries, e.g. Sen (1981, 1985), Dasgupta and Weale (1992), Dasgupta (1993) and Kakwani (1993). Kakwani (1993) rigorously formulated properties for an index of this type and suggested a particular index. Since well-being of a population depends on a bundle of attributes and people do not separate different aspects of their lives, for aggregating increments in different attributes, it is necessary to construct a multidimensional index of improvement. Tsui (1996) extended Kakwani's set up to a multidimensional structure and axiomatically characterized a multi-attribute index.

The functional form of an index of well-being improvement depends largely upon what we want to know about well-being. One has to first set up the purpose of measurement and then find a suitable measure within the framework. For instance, one may argue that in addition to measuring the extent of increase in well-being, an improvement index should also reflect the view that the marginal social valuation of (say) an additional year of life should depend on income. All suitably designed nonadditive

† An earlier draft of this paper was presented at the Far Eastern Meeting of the Econometric Society held at the Chinese University of Hong Kong in July, 1997. The authors wish to thank the participants for their comments. They also thank an anonymous referee for his comments and suggestions.

indices (including the Tsui index), which are not arithmetic average of attribute-wise indices, should serve this purpose.[1]

Given that in calculating human development index, UNDP (1991–98) attaches equal importance to all attributes of well-being and that improvement refers to gain in achievement which reflects human development, another objective of an improvement index can be the determination of percentage contributions made by different attributes to the overall improvement.[2] This will enable a policy marker to identify the indicators whose contributions are rather low or negative and recommend policies under which more resources can be allocated for improving the levels of these indicators of well-being. Clearly, an additive improvement index which can be expressed as the average of improvement indices for individual attributes, will help us to carry out this type of analysis. We may observe here that, according to this notion of policy recommendation, an assessment of overall progress becomes contingent on the implicit valuation of our index. However, it may be useful to do this for two reasons. First, following Sen (1985), the nonwelfarist approach to policy analysis is becoming quite popular. Second, often policy is evaluated by the use of such indices. Therefore, it seems worthwhile to investigate what kind of policy would be implied by using a particular index of improvement. It should be evident that a nonadditive improvement index cannot be employed to do this kind of exercise.

Obviously, an additive index shows that the valuation attached to an extra unit of an attribute does not depend on another variable. But such dependences are not always meaningful. For instance, it is unlikely that the marginal social valuation of an extra unit of a public good (e.g. national highway) will depend on infant survival rate. Even if it is possible to talk about such relationships, empirical literature does not always support them strongly. For example, there has been a debate about the importance of low incomes as a determinant of undernutrition (see Lipton and Ravallion, 1995).

Thus, given that additive and nonadditive measures may serve two different purposes in well-being improvement measurement, it seems worthwhile to make a detailed analysis of these measures. This is the purpose of this paper. In the next section of the paper we present the properties for a measure of improvement in well-being. The entire class of additive improvement indices is characterized in section 3. In a particular case one member of this family becomes the difference between the UNDP (1991–98) human development indices for the periods under consideration. Section 4 introduces a family of nonadditive improvement indices. The Tsui (1996) index turns out to be a member of

[1] It may be noted that Tsui (1996) did not impose this requirement as a postulate for an index of improvement.

[2] UNDP (1991–98) defined human development index as the unweighted arithmetic average of normalized values of life expectancy at birth, educational attainment and per capita real GDP. Though high correlation has been detected among these three variables, principal component analysis carried out by UNDP shows that the eigenvector corresponding to the leading eigenvalue (which explains 88% of the total variance in data) puts virtually equal weight on the three variables. 'Thus it does not advocate omitting or downgrading a variable' (UNDP, 1993, p. 119). Noorbakhsh (1998) defined an alternative human development index in terms of the Euclidean distance of the standardized attribute levels from the respective highest quantities. As in the case of the UNDP index, this modified index also regards all the attributes as equally important.

this family. A numerical illustration of both additive and nonadditive indices is provided in section 5. Finally, section 6 concludes.

2. PROPERTIES FOR A MEASURE OF IMPROVEMENT IN WELL-BEING

Suppose that there are n attributes of well-being. An improvement in well-being with respect to the i^{th} attribute can be conceptualized as an increase in the attainment level of the attribute from one value to another. Let x_{it} stand for the value of attribute i in period t. Then $x_t = (x_{1t}, x_{2t}, \dots, x_{nt})$ is the vector of attributes in period t. Suppose that we wish to determine the improvement in well-being between periods 1 and 2. Then an improvement index showing the extent of improvement in people's welfare between these two periods should be a real valued function of x_1 and x_2.

An alternative way of viewing improvement is in terms of reduction in absolute or normalized deprivations. To see this, let M_i be the upper bound of the i^{th} attribute and m_i be its lower bound. (See Morris, 1979, Sen, 1981, Dasgupta, 1993 and Dasgupta and Weale, 1992 for discussion on such bounds.) Thus, $x_{it} \in [m_i, M_i]$ for $i = 1, 2, \dots, n$ and $t = 1, 2$. We assume that $m_i < M_i$. (This assumption is implicit in Kakwani, 1993, Tsui, 1996 and Majumder and Chakravarty, 1996, 1996a.) This assumption ensures that the open set (m_i, M_i) is nonempty. Let $M = (M_1, M_2, \dots, M_n)$ and $m = (m_1, m_2, \dots, m_n)$. The shortfall of the value of the attribute from its maximum attainable value $M_i - x_{it}$ is the deprivation with respect to this attribute. The smaller is this shortfall the better off the society is with respect to the attribute under consideration. The improvement index then becomes a real valued function of $(M_1 - x_{11}, M_2 - x_{21}, \dots, M_n - x_{n1})$ and $(M_1 - x_{12}, M_2 - x_{22}, \dots, M_n - x_{n2})$. We can as well regard improvement index as a real valued function of $(d_{11}, \dots, d_{n1})$ and $(d_{12}, \dots, d_{n2})$ where d_{it}, more precisely, $d_{it}(M_i, m_i, x_{it}) = (M_i - x_{it})/(M_i - m_i)$ for all $i = 1, 2, \dots, n$ and $t = 1, 2$. Since $x_{it} \geq m_i$ for all i and t, d_{it} is the deprivation with respect to attribute i in period t, expressed as a proportion of its maximal attainable value. Clearly, d_{it} is normalized over the set [0, 1], that is, $d_{it} \in [0, 1]$. Let $d_t = (d_{1t}, \dots, d_{nt})$ where $t = 1, 2$. Since human development indicators (e.g. UNDP, 1993) and the Kakwani and Tsui measures of improvement are based on normalized deprivations, we will also define the improvement index directly on normalized deprivations. That is, an improvement index is a real valued function Q, which associates to any vector of normalized deprivations d_1 and d_2 in periods 1 and 2, a value $Q(d_1; d_2)$ indicating the level of improvement that actually takes place when normalized deprivation changes from d_1 to d_2. We will show later that improvement indices defined this way satisfy a desirable property.

We now suggest some postulates for an arbitrary Q. The first property is regarding the domain of Q. Since $d_{it} \in [0, 1]$, $d_i \in [0, 1]^n$, where $[0, 1]^n$ is the n-fold cartesian product of [0, 1]. Thus, we have:

Domain Restriction (DR): Q is a real valued function defined on $[0, 1]^n \times [0, 1]^n$. More precisely, $Q : [0, 1]^n \times [0, 1]^n \to R^1$, where R^1 is the real line.

We may point out that Tsui (1996) adopted the absolute shortfall domain $\prod_{i=1}^{n} D^i \times \prod_{i=1}^{n} D^i$, where $D^i = (0, M_i - m_i]$, instead of our normalized domain. However, the final form of the Tsui index, which is derived axiomatically, is shown to depend on normalized deprivations which are elements of our normalized domain.[3]

The next six postulates which, excepting the first one, are straight generalizations of the corresponding Kakwani postulates, have been suggested by Tsui (1996). Since Q is defined on the normalized domain instead of the absolute domain considered by Tsui (1996), here we state these properties under appropriate modifications.

It is desirable that minor changes in the arguments of Q should not give rise to abrupt changes in Q. That is, Q should satisfy continuity.

Continuity (CN): Q is a continuous function.

The deprivation level d_{i2} rises only if x_{i2} falls. Thus, Q should be negatively related to d_2. Similarly, Q should be positively related to d_1.

Monotonocity (MN): Q is an increasing function of d_1 and a decreasing function of d_2.

Kakwani (1993) argued that for any three periods 1, 2 and 3, the improvement from period 1 to period 3 can be expressed as the sum of improvements from periods 1 to 2 and that from periods 2 to 3. This shows how comparisons in well-being between any two periods can be expressed as the sum of comparisons for many intermediate periods. Since the total change is the sum of changes for intermediate periods, the problem of transitivity does not arise at all. This property is formally stated in the present framework as

Period Consistency (PC): For any $x_t \in \prod_{i=1}^{n}[m_i, M_i]$, $t = 1, 2, 3$; $Q(d_1; d_3) = Q(d_1; d_2) + Q(d_2; d_3)$.

It is customary to have a normalization condition. It says that the highest attainable value of Q is 1. The highest value of Q is attained when the increases in the values of the attributes are maximum, that is, when $M_i = x_{i2}$ and $m_i = x_{i1}$ for all i. In other words, improvement is maximized when all the normalized deprivations in period 1 are unity and in period 2 are zero.

Normalization (NR): $Q(d_1; d_2) = 1$ when $d_2 = 01^n$ and $d_1 = 1^n$, where 1^n is n-coordinated vector of ones.

Sen (1981) argued that for certain attributes of well-being, such as life expectancy, it becomes harder to increase improvement at higher achievement levels of the attribute (see also Sen, 1992, Dasgupta, 1993 and Kakwani, 1993). In multidimensional framework, this requirement can be stated rigorously as

Increasing Difficulty of Improvement (ID): For any x_1, z_1 such that $x_{j1} = z_{j1}$ for all $j \neq i$ and $x_{i1} > z_{i1}$,

$$Q(d^{x_1}; d^{x_i + ce_i}) > Q(d^{z_1}; d^{z_1 + ce_i})$$

where

$$d^{x_1} = \left(\frac{M_1 - x_{11}}{M_1 - m_1}, \ldots, \frac{M_n - x_n}{M_n - m_n}\right),$$

[3] Tsui (1996) provides discussions on alternative domain assumptions.

and $d^{x_1+ce_i}$ is the normalized deprivation vector all of whose components, except the i^{th}, are same as those of d^{x_1} and the i^{th} component of $d^{x_1+ce_i}$ is $(M_i - x_{i1} - c)/(M_i - m_i)$, with $c > 0$ being any arbitrary constant such that $x_{i1} + c < M_i$. The vectors d^{z_1} and $d^{z_1+ce_i}$ are defined analogously.

The next property is concerning the comparability of improvement levels of two populations.

Full Comparability (FC): For any $x_1, x_2, y_1 \in \prod_{i=1}^{n}[m_i, M_i)$, there always exists some $y_2 \in \prod_{i=1}^{n}[m_i, M_i)$ such that

$$Q(d^{x_1}; d^{x_2}) = Q(d^{y_1}; d^{y_2})$$

where d^{x_1} is the normalized deprivation vector based on the achievement vector x_1, and so on. FC requires that it is always possible for a country with achievement level y_1 to attain some achievement level y_2 so that the degree of improvement becomes equal to that of another country with the achievement vectors x_1 and x_2.

The final property is the dimensionality (DM) postulate. According to DM the improvement index should be insensitive to the units of measurement of the attributes. For instance, if longevity is measured in months instead of in years, then improvement should not change.

Dimensionality (DM): For any $x_t \in \prod_{i=1}^{n}[m_i, M_i]$, $t = 1, 2$,

$$\begin{aligned} Q(d_{11}(\alpha_1 M_1, \alpha_1 m_1, \alpha_1 x_{11}), d_{21}(\alpha_2 M_2, \alpha_2 m_2, \alpha_2 x_{21}), \ldots, d_{n1}(\alpha_n M_n, \alpha_n m_n, \alpha_n x_{n1});\\ d_{12}(\alpha_1 M_1, \alpha_1 m_1, \alpha_1 x_{12}), d_{22}(\alpha_2 M_2, \alpha_2 m_2, \alpha_2 x_{22}), \ldots, d_{n2}(\alpha_n M_n, \alpha_n m_n, \alpha_n x_{n2}))\\ = Q(d_{11}(M_1, m_1, x_{11}), d_{21}(M_2, m_2, x_{21}), \ldots, d_{n1}(M_n, m_n, x_{n1});\\ d_{12}(M_1, m_1, x_{12}), d_{22}(M_2, m_2, x_{22}), \ldots, d_{n2}(M_n, m_n, x_{n2})). \end{aligned}$$

where $\alpha_i > 0$ is any scaler.

Since each d_{it} is homogeneous of degree zero in its arguments, that is, $d_{it}(\alpha_i M_i, \alpha_i m_i, \alpha_i x_{it}) = d_{it}(M_i, m_i, x_{it})$ for all positive α_i, DM is always satisfied by an improvement index defined on normalized deprivation levels. Majumder and Chakravarty (1996) employed DM in the single-dimensional case to characterize the Kakwani index of improvement. Tsui (1996) adopted an analogous property, which he called homotheticity (HM), for pinning down a specific class of indices. According to HM, the ranking of a pair of shortfalls $(M - x_1, M - x_2)$ and $(M - y_1, M - y_2)$ remains unchanged if, for each i, $(M_i - x_{it})$ and $(M_i - y_{it})$ are multiplied by some positive scaler. Following Tsui (1996) we can argue that there does not exist a Q that will satisfy the properties DR, CN, MN, PC, FC, ID, NR and HM simultaneously. If some Q has to be designed, one of the properties has to be given up. Kakwani noted some problems with FC. In the single-dimensional case, Majumder and Chakravarty (1996) developed a sufficient condition under which FC can be fulfilled. Given the problems associated with FC, we will give up FC and not impose it as a postulate for Q.

3. THE FAMILY OF ADDITIVE IMPROVEMENT INDICES

The purpose of this section is to characterize the class of additive improvement indices. An improvement index Q is called additive across its components if it satisfies the following postulate.

Additivity (AD): For any $w_t \in \prod_{i=1}^{k}[m_i, M_i]$, $i = 1, 2$;

$$Q(d_1; d_2) = \frac{1}{k}\sum_{i=1}^{k} Q^i(d_{i1}, d_{i2}),$$

where $Q^i : [0, 1] \times [0, 1] \to R^1$ is the improvement index based on attribute i only. This property says that the overall improvement level is simply the arithmetic average of improvement indices based on individual attributes. This subdivision allows qualitative as well as quantitative assessment of attribute—wise improvement. The quantity $T_i = Q^i(d_{i1}, d_{i2})/k$ may be interpreted as the total contribution for attribute i to overall improvement Q, while $100T_i/Q$ is the percentage contribution of attribute i. Therefore, this type of breakdown will help us to isolate the attributes which are less susceptible to aggregate level of improvement.

To identify the general family of additive improvement measures, let us consider the class F of all real valued increasing functions defined on $[0, 1]$ which are strictly concave, continuous on $(0, 1]$ and for which the difference between the functional values at 1 and 0 is 1. More precisely, $f : [0, 1] \to R^1$ is a member of F if f is increasing on the entire domain, strictly concave and continuous on the subdomain $(0, 1]$ and $f(1) - f(0) = 1$.

Examples of functions which are members of F are

(i) $f_1(t) = t^r, 0 < r < 1$.

(ii) $f_2(t) = (1 - e^{-t})/(1 - e^{-1})$.

(iii) $f_3(t) = 2t/(1 + t)$.

Note that, for the condition $f(1) - f(0) = 1$ to hold, we do not require $f(1) = 1$ and $f(0) = 0$. For instance, in the above examples, if we define $g_i(t) = f_i(t) + \alpha_i$, where α_i is a constant, $i = 1, 2, 3$; then g_i's are also members of F.

In the theorem proved below we characterize the class of additive improvement indices in terms of members of F.

THEOREM 1. *An index of improvement in well-being* $Q : [0, 1]^k \times [0, 1]^k \to R^1$ *satisfies* MN, NR, PC, ID *and* AD *if and only if*

$$Q(d_1, d_2) = \frac{1}{k}\sum_{i=1}^{k}(f_i(d_{i1}) - f_i(d_{i2})), \tag{1}$$

where $f_i \in F$ *for all* $i = 1, 2, \ldots, k$.

Proof. By additivity we can write Q as

$$Q(d_1, d_2) = \frac{1}{k}\sum_{j=1}^{k} Q^j(d_{j1}, d_{j2}). \tag{2}$$

But by PC, for any $d_j \in [0, 1]^k$, where $j = 1, 2, 3$,

$$Q(d_1; d_2) = Q(d_1; d_3) + Q(d_3; d_2) . \tag{3}$$

Therefore, AD combined with PC gives

$$Q(d_1, d_2) = \frac{1}{k} \sum_{j=1}^{k} (Q^j(d_{j1}, d_{j3}) + Q^j(d_{j3}, d_{j2})) . \tag{4}$$

The equation

$$Q^j(d_{j1}, d_{j3}) + Q^j(d_{j3}, d_{j2}) = Q^j(d_{j1}, d_{j2})$$

is a linear functional equation whose only solution is

$$Q^j(d_{j1}, d_{j2}) = f_j(d_{j1}) - f_j(d_{j2}) , \tag{5}$$

where $f_j : [0, 1] \to R^1$ (see Aczél, 1966, p. 232). Thus (4) can be rewritten as

$$Q(d_1, d_2) = \frac{1}{k} \sum_{j=1}^{k} (f_j(d_{j1}) - f_j(d_{j2})) . \tag{6}$$

Clearly, MN requires increasingness of f_i over [0, 1]. Now, suppose $d_{i1} = 1$ and $d_{i2} = 0$ for all i. Then Q given by (6) becomes

$$Q(d_1, d_2) = \frac{1}{k} \sum_{j=1}^{k} (f_j(1) - f_j(0)) . \tag{7}$$

But by NR, in this extreme case

$$Q(d_1, d_2) = 1 . \tag{8}$$

From (7) and (8) we have

$$\sum_{j=1}^{k} (f_j(1) - f_j(0)) = k . \tag{9}$$

Note that, by increasingness of f_j, $f_j(1) - f_j(0)$ is positive for all j. Now, (9) is true for all $k \geq 1$. Therefore, for $k = 1$, we have $f_1(1) - f_1(0) = 1$. This shows that $f_j(1) - f_j(0) = 1$ for all j.

Following Tsui (1996, p. 294) we can say that if Q satisfies ID, then f_i will be strictly concave on (0, 1]. This in turn implies that f_i is continuous in the interior of (0, 1], that is, on (0, 1). Hence f_i is left continuous at 1. But for $f_i \in F$, we have to show that f_i is continuous on (0, 1]. So, now it is necessary to demonstrate right continuity of f_i at 1.

To show right continuity of f_i at 1, we note that if $y_i \in (0, 1)$, then there exists a positive integer $N \geq 1$ such that $n > N$ will imply $y_i + \frac{y_i}{n} \leq 1$, $y_i - \frac{y_i}{n} > 0$. Therefore by ID

$$0 < f_i\left(y_i + \frac{y_i}{n}\right) - f_i(y_i) < f_i(y_i) - f_i\left(y_i - \frac{y_i}{n}\right) . \tag{10}$$

Taking limit as $n \to \infty$ in (10), we have

$$0 \leq f_i(y_i+) - f_i(y_i) \leq f_i(y_i) - f_i(y_i-) . \tag{11}$$

But by left continuity of f_i, $f_i(y_i) - f_i(y_i-) = 0$. Therefore, from (11) it follows that $f_i(y_i) = f_i(y_i+)$, which means that f_i is right continuous on (0, 1]. Hence we have right continuity of f_i at 1. This establishes the necessity part of the theorem.

To prove sufficiency, we observe that increasingness of f_i ensures MN and the condition $f_i(1) - f_i(0) = 1$ guarantees NR. AD and PC are obviously satisfied by (1). To show that strict concavity of f_i is sufficient for ID, take any $0 < t < s < v \leq 1$. Define $\lambda = (v - s)/(v - t)$. Then $s = \lambda t + (1 - \lambda)v$. By strict concavity of f_i,

$$f_i(\lambda t + (1 - \lambda)v) > \lambda f_i(t) + (1 - \lambda) f_i(v)\,, \tag{12}$$

or,

$$f_i(s) > \frac{v - s}{v - t} f_i(t) + \frac{s - t}{v - t} f_i(v)\,, \tag{13}$$

which on rearrangement gives

$$[f_i(v) - f_i(s)](s - t) < [f_i(s) - f_i(t)](v - s)\,, \tag{14}$$

that is,

$$\frac{f_i(v) - f_i(s)}{v - s} < \frac{f_i(s) - f_i(t)}{s - t}\,. \tag{15}$$

Similarly, for $0 < t < s < v < r \leq 1$, we have

$$\frac{f_i(s) - f_i(t)}{s - t} > \frac{f_i(v) - f_i(s)}{v - s} > \frac{f_i(r) - f_i(v)}{r - v}\,. \tag{16}$$

Now choose $w_1, v_1 \in \prod_{j=1}^{n}[m_j, M_j)$ as described in the postulate ID. Then

$$Q(d^{w_1}; d^{w_1 + ce_i}) = \frac{1}{k}\left(f_i\left(\frac{M_i - w_{i1}}{M_i - m_i}\right) - f_i\left(\frac{M_i - w_{i1} - c}{M_i - m_i}\right)\right) \tag{17}$$

and

$$Q(d^{u_1}; d^{u_1 + ce_i}) = \frac{1}{k}\left(f_i\left(\frac{M_i - u_{i1}}{M_i - m_i}\right) - f_i\left(\frac{M_i - u_{i1} - c}{M_i - m_i}\right)\right)\,. \tag{18}$$

From (17) and (18), $Q(d^{w_1}; d^{w_1+ce_i}) > Q(d^{u_1}; d^{u_1+ce_i})$ means that

$$f_i\left(\frac{M_i - w_{i1}}{M_i - m_i}\right) - f_i\left(\frac{M_i - w_{i1} - c}{M_i - m_i}\right) > f_i\left(\frac{M_i - u_{i1}}{M_i - m_i}\right) - f_i\left(\frac{M_i - u_{i1} - c}{M_i - m_i}\right)\,. \tag{19}$$

Letting $x_i = (M_i - w_{i1})/(M_i - m_i)$, $y_i = (M_i - u_{i1})/(M_i - m_i)$ and $\alpha_i = c/(M_i - m_i)$, inequality (19) becomes

$$f_i(x_i) - f_i(x_i - \alpha_i) > f_i(y_i) - f_i(y_i - \alpha_i)\,. \tag{20}$$

Now, to demonstrate ID, we have to show (20) for any $x_i < y_i$ and $\alpha_i > 0$. Suppose $\alpha_i > 0$ is such that $x_i + \alpha_i < y_i$. Then take $t = x_i$, $s = x_i + \alpha_i$, $v = y_i$, $r = y_i + \alpha_i$ in (16) to get (20). If, on the other hand, $\alpha_i > 0$ is such that $x_i + \alpha_i > y_i$, take $t = x_i$, $s = y_i$, $v = x_i + \alpha_i$, $r = y_i + \alpha_i$ in (16) to get

$$f_i(y_i) - f_i(x_i) > f_i(y_i + \alpha_i) - f_i(x_i + \alpha_i)\,.$$

Now, by transposing $f_i(y_i)$ and $f_i(x_i + \alpha_i)$ we get (20). This completes the proof of the theorem. ■

The proof of the theorem shows that ID can be regarded as a sufficient condition for continuity of f_i from which continuity of Q on $(0, 1]^k \times (0, 1]^k$ follows. Evidently, by construction Q satisfies DM.

To illustrate the general formula Q, let us suppose for simplicity that f_i's are identical, that is $f_i = f$ for all i and the function f is of the type $f(t) = t^r, 0 < r < 1$. Then Q becomes

$$\begin{aligned} Q_r &= \frac{1}{k}\sum_{i=1}^{k} Q_r(d_{i1}, d_{i2}) \\ &= \frac{1}{k}\sum_{i=1}^{k}(d_{i1}^r - d_{i2}^r)\,. \end{aligned} \tag{21}$$

The parameter r reflects different perceptions of improvement. If $w_{i1} < w_{i2}$ for all i, that is, if improvement takes place with respect to all attributes, then Q_r increases as r increases. In this case a higher value of r gives greater emphasis to the attributes whose contributions are low to the overall improvement. However, if the inequality $w_{i1} < w_{i2}$ is violated for some i, then nothing can be concluded unambiguously about the monotonocity of Q_r with respect to r. On the other hand, if $w_{i1} > w_{i2}$ for all i, that is, if there has not been improvement with respect to any attribute, then Q_r decreases as r increases.

For $r = 0$, $Q_r = 0$. In contrast, for $r = 1$, $Q_r = \frac{1}{k}\sum_{i=1}^{k}\left(\frac{w_{i2}-w_{i1}}{M_i-m_i}\right)$, the average of increase in the attainment levels of the attributes, expressed as fraction of maximum increase in attainment levels. Note that when $r = 1$, Q_r can also be written as

$$\begin{aligned} Q_1 &= \frac{1}{k}\sum_{i=1}^{k} d_{i2} - \frac{1}{k}\sum_{i=1}^{k} d_{i1} \\ &= H_2 - H_1\,, \end{aligned} \tag{21}$$

where H_i is the UNDP (1991–98) human development index for period i, $i = 1, 2$. Thus, Q_1 is simply the difference between the human development indices for the periods considered. However, Q_1 violates the postulate ID.

If there is only one attribute, $Q_r = d_{i1}^r - d_{i2}^r$, the Kakwani index of improvement, which is also the Tsui index for the single attribute case.[4] For $r = 1$, in the single attribute case, Q_r is related to the Sen (1981) index S by $Q_r = S\frac{M_1-m_1}{M_1-w_{11}}$.

It is clear that, given $f_i \in F$, there exists a corresponding improvement index Q. These indices will differ only in the manner how we transform the deprivation levels d_{i1} and d_{i2} into values $f_i(d_{i1})$ and $f_i(d_{i2})$ using the transformations f_i. However, for $f_i \in F$, the index Q will meet all the desirable properties of an improvement index.

[4] Kakwani (1993) also argues that $\log(d_{11}) - \log(d_{12})$ can be regarded as an improvement index. But because of some shortcoming pointed out by Tsui for this index, we will not go for further analysis of this form.

We may mention here that the Tsui multidimensional index, which is given by

$$U = \frac{\prod_{i=1}^{k}(M_i - w_{i1})^{r_i} - \prod_{i=1}^{k}(M_i - w_{i2})^{r_i}}{\prod_{i=1}^{k}(M_i - m_i)^{r_i}}, \quad 0 < r_i < 1, \tag{23}$$

meets all the postulates except additivity. However, Tsui's major objective was to generalize the Kakwani index to a multidimensional framework, which he did quite successfully. It may be noted that U is not monotonic in r_i.

The following remark, which may be regarded as a supplementary remark to Corollary 3 of Tsui (1996), shows that the properties MN, NR, PC, ID and AD are independent. Demonstration of independence involves the construction of an index of improvement that will satisfy four of the five properties but not the remaining one.

REMARK 1. Properties MN, NR, PC, ID and AD are independent.

Proof. (a) Since

$$Q^1 = \frac{1}{k(1-e)} \sum_{i=1}^{k} (e^{d_{i2}} - e^{d_{i1}}) \tag{24}$$

is increasing in d_{i2} and decreasing in d_{i1}, it violates MN. However, it meets all the other four properties.

(b) The index given by

$$Q^2 = \frac{\alpha}{k} \sum_{i=1}^{k} ((d_{i1})^r - (d_{i2})^r), \tag{25}$$

where $0 < r < 1$ and $\alpha > 1$, fulfills all the properties except NR (since $\alpha \neq 1$).

(c) Because of linearity in d_{i1} and d_{i2}, the index

$$Q^3 = \frac{1}{k} \sum_{i=1}^{k} (d_{i1} - d_{i2}) \tag{26}$$

fails to satisfy ID. But it satisfies the other postulates.

(d) Since

$$Q^4 = \frac{1}{k} \sum_{i=1}^{k} \left(\frac{1 - d_{i2}}{2 - d_{i1}} \right) \tag{27}$$

is not separable in d_{i1} and d_{i2}, it is not period consistent. However, it is monotone, normalized, additive and shows increasing difficulty in improvement at higher attainment levels.

(e) We have already mentioned that the Tsui index given by (23) meets all the properties except AD. ■

4. A FAMILY OF NONADDITIVE IMPROVEMENT INDICES

Given that additive and nonadditive measures of improvement have their own merits, it may be worthwhile to consider nonadditive measures also. As a general nonadditive measure of improvement in well-being, we suggest the use of

$$T(d_1, d_2) = \prod_{j=1}^{k} f_j(d_{j1}) - \prod_{j=1}^{k} f_j(d_{j2}), \tag{28}$$

where $f_j : [0, 1] \to R^1$ is increasing, strictly concave and continuous on $(0, 1]$, $f(0) = 0$ and $f(1) = 1$. Note that the assumptions $f(0) = 0$ and $f(1) = 1$ are sufficient for the normalization rule to hold. If we denote the class of such real valued functions defined on $[0, 1]$ by $\bar{F}$, then evidently $\bar{F} \subset F$.

The following remark, whose proof is easy, shows that T meets all the properties for a measure of improvement.

REMARK 2. The general measure T introduced in (28) satisfies MN, NR, PC, ID, CN and DM.

In order to illustrate T, let us suppose that $f_j(t) = t^{r_j}$, $0 < r_j < 1$. Then it is evident that the resulting index becomes the Tsui index of improvement given by (23). Thus, given that any member of the family $\bar{F}$ generates a satisfactory measure of nonadditive improvement index, the measure T can be regarded as a generalization of the Tsui index. The choice of particular functional forms $f_j \in \bar{F}$ for aggregating the improvement levels of alternative indicators into an overall measure is essentially a matter of value judgment. One minor shortcoming of the general index T (and hence also of the Tsui index) is that if $d_{ji} = 0$ for some j, i; then the product $\prod_{j=1}^{k} f_j(d_{ji})$ will be zero even if the remaining d_{ji} values are nonzero. This in turn ignores the effect of attributes with nonzero deprivation in the aggregation.

5. NUMERICAL ILLUSTRATION

In this section the index Q_r and the Tsui index are applied to UNDP data on human development to illustrate the usefulness of additivity. For illustrating the Tsui index also, we assume that $r_j = r$ and denote it by U_r. The UNDP reports provide data on several attributes of well-being for more than 170 countries. Depending on the values of different attributes of well-being, these reports subdivide the countries into three groups: high human development (HHD), medium human development (MHD) and low human development (LHD).

For our analysis we choose six countries. The countries chosen are Canada, France and Romania (HHD), Saudi-Arabia (MHD), and India and Rwanda (LHD). The periods over which we look at change in well-being is the interval 1987–1995. In fact, given the UNDP data, this is the longest period that can be covered. However, for Rwanda we concentrate on the period 1987–1993, since the relevant data for this country were not available for later years. The attributes of well-being we take for our analysis are life expectancy at birth, e (in years); adult literacy, l (in percentage) and real GDP per capita, g (in purchasing power parity \$). Although here we are looking at changes in well-being between the periods 1987 and 1995, we can consider any intermediate period

t between these two years. By period consistency, the sum of improvements from 1987 to t and then from t to 1995 is same as that from 1987 to 1995.

For each attribute the maximum value is taken as the maximum observed value of the attribute over countries and over two time periods considered. More precisely, let x_{it}^c be the observed value of the attribute i for country c at time t where $i = e, l, g$ and $t =$ 1987, 1995. Then $M_i = \max_{c,t} x_{it}^c$. We choose the minimum values in a similar way. That is, $m_i = \min_{c,t} x_{it}^c$. The maximum and minimum values of different attributes and the corresponding countries and year for which they are observed are $M_e = 79.9$ (Japan, 1995), $M_l = 99.0$ (many countries including Canada, France, USA belonging to the HHD, 1995 and also many countries in HHD group, 1987), $M_g = 34004$ (Luxembourg, 1995), $m_e = 34.7$ (Sierra Leone, 1995), $m_l = 12$ (Somalia, 1987), $m_g = 400$ (Chad, 1987). (It may be noted that these minimum and maximum values are not affected by 1993 values of the variables. Thus, use of 1993 data for Rwanda does not change these upper and lower limits.)

The relationship between well-being improvement and specific factors of improvement may be analyzed with the aid of a collection of tables called an improvement profile. Tables 1–4 shows such a profile, describing improvement in well-being generated by different sources. In Table 1, the first column gives the names of the countries for which the analysis is carried out. In columns 2–4 we present, for each country, the level of improvement for the three sources of well-being assuming that r =.25. These improvement levels are then averaged to determine the overall improvement $Q_{.25}$ which

Table 1. $U_{.25}$ and Subdivision of Improvement $Q_{.25}$ in Well-being by Sources of Improvement.

Country	Improvement based on			$Q_{.25}$	Percentage contribution to $Q_{.25}$ based on			$U_{.25}$
	e	l	g		e	l	g	
Canada	.1385	0	.0766	0.0717	64.4	0	35.6	0.1459*
France	.0670	0	.1127	0.0599	37.3	0	62.7	0.1124*
Saudi-Arabia	.0985	.0413	.0018	0.0472	69.6	29.2	1.2	0.1045
Romania	−.0248	.1035	.0115	0.0301	−27.5	114.7	12.8	0.0622
India	.0269	.0384	.0028	0.0227	39.6	56.3	4.1	0.0564
Rwanda	−.0130	.0507	.0013	0.0130	−33.2	130.0	3.2	0.0363

Table 2. $U_{.5}$ and Subdivision of Improvement $Q_{.5}$ in Well-being by Sources of Improvement.

Country	Improvement based on			$Q_{.5}$	Percentage contribution to $Q_{.5}$ based on			$U_{.5}$
	e	l	g		e	l	g	
Canada	.1203	0	.1245	0.0816	49.1	0	50.9	0.1037*
France	.0682	0	.1853	0.0845	26.9	0	73.1	0.0945*
Saudi-Arabia	.1419	.0679	.0033	0.0711	66.6	31.8	1.6	0.1160
Romania	−.0336	.0785	.0224	0.0224	−50.0	116.6	33.4	0.0311
India	.0437	.0673	.0056	0.0389	37.5	57.7	4.8	0.0797
Rwanda	−.0237	.0866	.0025	0.0218	−36.3	132.4	3.9	0.0567

Table 3. $U_{.75}$ and Subdivision of Improvement $Q_{.75}$ in Well-being by Sources of Improvement.

Country	Improvement based on			$Q_{.75}$	Percentage contribution to $Q_{.75}$ based on			$U_{.75}$
	e	l	g		e	l	g	
Canada	.0790	0	.1519	0.0770	34.2	0	65.8	0.0560*
France	.0521	0	.2290	0.0937	18.5	0	81.5	0.0599*
Saudi-Arabia	.1537	.0838	.0047	0.0807	63.5	34.6	1.9	0.0969
Romania	−.0342	.0449	.0328	0.0145	−78.7	103.3	75.4	0.0117
India	.0532	.0885	.0083	0.0500	35.5	59.0	5.5	0.0846
Rwanda	−.0326	.1110	.0038	0.0274	−39.7	135.1	4.6	0.0663

Table 4. U_1 and Subdivision of Improvement Q_1 in Well-being by Sources of Improvement.

Country	Improvement based on			Q_1	Percentage contribution to Q_1 based on			U_1
	e	l	g		e	l	g	
Canada	.0465	0	.1649	0.0705	22.0	0	78.0	0.0273*
France	.0354	0	.2519	0.0958	12.3	0	87.7	0.0339*
Saudi-Arabia	.1482	.0920	.0058	0.0820	60.2	37.4	2.4	0.0721
Romania	−.0310	.0230	.0426	0.0115	−89.5	66.4	123.1	0.0040
India	.0575	.1034	.0110	0.0573	33.5	60.2	6.4	0.0798
Rwanda	−.0398	.1264	.0050	0.0305	−43.5	138.0	5.5	0.0690

is presented in column 5. Columns 6–8 show, for each country, the percentage contributions of improvement with respect to the alternative attributes of well-being to overall improvement. Finally, the value of U_r (for $r = .25$) is reported in column 9. [In calculating these values, if the deprivation for some attribute is found to be zero, then we ignore it and consider only the nonzero deprivation levels. These figures are indicated by $*$ in the table.]

From Table 1 several interesting features emerge. We first analyze figures based on $Q_{.25}$. Although for each country, the overall level of improvement turns out to be positive, the picture appears to be dismal for some specific attributes for some countries. For instance, for Rwanda, life expectancy at birth has decreased significantly during the period 1987–1993. The percentage contribution to overall well-being improvement with respect to improvement in this factor is significantly negative (−33.2%). A very high adult literacy improvement makes the overall improvement index positive. (The contribution of the third factor is rather low as compared to these two factors.) On the other hand, for India we see that all the sources contribute positively to overall improvement, though the contribution of adult literacy is higher compared to that of the other two attributes. For Romania a high negative contribution comes from life expectancy at birth. This significant negative contribution outweighs the extremely high positive effect of literacy rate and makes the overall index quite close to that for India. It may be interesting to note that at the base period (1987) Romania belonged to the group HHD but low improvement during the period 1987–1995 relegated it to the group MHD in 1995. However, the groups of attachment of all other countries remained unaltered. We also note that, for India, the variation among the attributes in relation to their percentage contributions to overall improvement is minimum among the countries considered. However, the contribution of real GDP is rather low compared to that of other factors. This kind of breakdown has been observed for Saudi Arabia also. Among the countries considered, the maximum improvement during the reference period has been observed for Canada. For Canada and France adult literacy remained at the maximal level in both 1987 and 1995, but while for the former life expectancy at birth has been found to contribute highly to overall improvement, for the latter real GDP played a similar role.

A comparison of figures in columns 5 and 9 of Table 1 show that the ranking of the countries by both $Q_{.25}$ and $U_{.25}$ are the same. Further, for all countries $U_{.25}$ turns out to be higher than $Q_{.25}$. Thus, the use of the same functional form $f(t) = t^{.25}$ in calculating Q and U shows some kind of consistency between them. However, this consistency is not expected in general, because differing aggregation procedures are employed to generate Q_r and U_r from observations on deprivation levels of alternative attributes.

Tables 2, 3 and 4 present similar figures for r =.5, .75 and 1. We can analyze these tables in a manner in which we analyzed Table 1. Tables 1–4 show that the index values as well as percentage contributions are sensitive to the value of r. As expected, in all these cases the ranking of countries by U_r and Q_r do not coincide. We have pointed out earlier that if positive improvement takes place with respect to all attributes, then Q_r increases as r increases. However, the general nonmonotonic behaviour of U_r with respect to r is maintained in this case also. These are confirmed by index values calculated for India and Saudi Arabia.

6. CONCLUDING REMARKS

Kakwani (1993) constructed a class of improvement indices of well-being satisfying certain desirable properties. Tsui (1996) interestingly extended Kakwani's analysis to a multidimensional set-up. In this paper we have characterized the family of additive multidimensional improvement indices, that is, the indices which can be expressed as the average of improvement levels for different attributes of well-being. An index showing this type of breakdown becomes quite important from policy point of view—sources for which improvement is negative or low can be identified. We have also suggested a family of nonadditive improvement indices of which the Tsui index is a particular case. Evidently, the nonadditive measures are not helpful for implementing the above notion of policy prescription. However, the nonadditive measures explicitly recognize the dependence of marginal social valuation of an extra unit of one variable on another (explanatory) variable, a property not fulfilled by additive measures. Thus, both additive and nonadditive measures have their respective usefulness. A numerical illustration of both measures is also provided in the paper.

REFERENCES

1. Aczél, J. (1966): Lectures on Functional Equations and their Applications, Academic Press, New York and London.
2. Dasgupta, P. (1993): An Inquiry into Well-being and Destitution, Oxford, Clarendon Press.
3. Dasgupta, P. and M. Weale (1992): On Measuring the Quality of Life, World Development, 20.
4. Kakwani, N. (1993): Performance in Living Standards: An International Comparison, Journal of Development Economics 41, 307–336.
5. Lipton, M. and M. Ravallion (1995): Poverty and Policy, in J. Behrman and T. N. Srinivasan (Ed.), Handbook of Development Economics, Vol. 3, North Holland.
6. Majumder, A. and S. R. Chakravarty (1996): Achievement and Improvement in Living Standards, Journal of Development Economics 50, 189–195.
7. Majunder, A. and S. R. Chakravarty (1996a): Achievement and Improvement in Living Standards—An Empirical analysis, Social Indicators Research 37, 269–280.
8. Morris, D. (1979): Measuring the Condition of the World poor: the Physical Quality of Life Index, World Bank, Washington DC.
9. Noorbakhsh, F. (1998): A Modified Human Development Index, World Development 26, 517–528.
10. Sen, A. K. (1981): Public Action and the Quality of Life in Developing Countries, Oxford Bulletin of Economics and Statistics 43, 287–319.
11. Sen, A. K. (1985): Commodities and Capabilities, Amsterdam: North Holland.
12. Sen, A. K. (1992): Inequality Reexamined, Clarendon Press, New York.
13. Tsui, K. (1996): A Note on Kakwani's Index of Improvement in Well-being, Social Choice and Welfare 13, 291–303.
14. UNDP (1991–1998): Human Development Report, Oxford University Press, New York.