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LABOUR EFFORTS, INTERPERSONAL COMPARISONS OF UTILITY AND OPTIMAL TAXATION

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Abstract: We study the structure of optimal taxation in an economy composed of individuals with the same characteristics except for their productive abilities. We compare the optimal outputs, consumptions and tax rates corresponding to Harsanyi, Nash and Rawls solutions. Some definite results are obtained when there are two types of consumers, and specific studies are made on proportional taxation. We will also give conditions under which the utility level of the more able individual is lower than that of the less able individual and argue that the more able individual have incentive to pretend to be less able if separate tax rates are applied. (JEL D63, H21)

1. INTRODUCTION

We utilize a model of a single product economy with different types of consumers who derive the same utility from the same amount of consumption but differs with respect to their productive abilities. Our aim is to compare optimal tax rules and allocations derived from different welfare criteria.

In particular, we are interested in comparing the utilitarian solution of the Harsanyi type, the Nash bargaining solution and the Rawls maxmin solution. This problem was investigated only by numerical examples in the standard literature which includes Mirrlees (1971), Atkinson and Stiglitz (1980, Chapter 13) and Stiglitz (1982, 1987).

Some definite results are established, under appropriate assumptions, for an economy with two types of consumers. It is demonstrated that, when the government can apply lump-sum taxation for each individual, the optimal amount of taxation and the tax rate for the more able (resp., the less able) individual are highest (resp., lowest) for the Harsanyi social welfare function, second to it for

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the Nash social welfare function and lowest (resp., highest) for the Rawls social welfare function. The optimal amounts of production and consumption for the more able (resp., less able) individual are arranged in the same (resp., the opposite) order (Proposition 2). We will show that a similar result holds when the government applies proportional tax rates (Proposition 6). We will also give conditions under which the utility levels of the more able individual is lower than that of the less able individual for the Harsanyi and the Nash social welfare functions (Propositions 1 and 5).

Finally, we will examine whether consumers have incentive to reveal their true incomes, when the government has incomplete information about the characteristics of individuals. We will show that, for each of the three social welfare functions, if the government applies separate tax rates for the two types of consumers, then the more able individual becomes better off by not revealing his ability (Propositions 7 and 8).

The organization of this paper is as follows. In Section 2, we present the basic model and introduce different welfare criteria. The comparison of the optimal tax rules derived from different welfare criteria is made in Sections 3 and 4, under the assumption that the government has complete information about the characteristics of each individual and can specify the allocation of each individual by applying separate tax rules. Behaviour of an individual under a given tax function and corresponding optimal tax rules are investigated in Sections 5 and 6. In Section 7, we suppose that the government cannot distinguish productive abilities of individuals, and investigate whether individuals have incentives to pretend to be more productive. In Appendices 1 and 2, we study the properties of the utility possibility set and evaluate the utility levels of the consumers at the three basic solutions.

2. THE MODEL

We consider an economy consisting of n individuals and set $N=\{1, 2, \cdots, n\}$. We assume that individual i has the utility function of the form

$$U_i=v(x_i)-c_i(y_i) \qquad (i\in N)\,, \tag{1}$$

where $x_i\in R$ denotes the amount of consumption and $y_i\in R$, the income of the consumer i ($i\in N$). It is implicit here that consumers derive the same enjoyment from the same level of consumption, but may derive different disutility to produce the same amount of output.[1] x_i and y_i are related as

$$x_i=y_i-T_i \qquad (i\in N)\,, \tag{2}$$

[1] This point may be explained more explicitly as follows. If (i) individual i's income y_i is expressed as a monotone increasing function $f_i(L_i)$ of his labor effort L_i and (ii) the utility function $U_i(x_i, y_i)$ is the same for all i and can be expressed as $v(x_i)-c(y_i)$ then we have a utility function of form (1) by eliminating L_i and defining $c_i(\cdot)$ in an appropriate way.

where $T_i \in R$ denotes the tax levied on individual i.

We first make the following assumption on the utility function.

ASSUMPTION 1. (i) $v'(x_i)>0$, $v''(x_i)<0$, $v(0)=0$

(ii) $c_i'(y_i)>0$, $c_i''(y_i)>0$, $c_i(0)=0$ *with* $c_i'(y)<c_j'(y)$ *for each* $i<j$ $(i, j \in N)$.

The last condition intends to mean, in particular, that if $i<j$, i needs less labor or less effort than j to produce the same amount of output. Assumption 1 (ii) implies that $c_i(y)<c_j(y)$ for each $i<j$ and $y>0$.

The set of attainable allocations A for the economy is defined as the set of n-tuples of consumption-production pairs $\{(x_i, y_i) \mid i \in N\}$ which satisfy

$$\sum_i x_i = \sum_i y_i \,. \tag{3}$$

We next proceed to consider how to choose the best points in the attainable set using different welfare criteria.

As one typical case, we consider the situation where all individuals have the knowledge of his and other people's utility functions. The Nash bargaining solution (Nash (1950)) $\{(x_i^*(0), y_i^*(0)) \mid i \in N\}$ is given as the allocation which attains the maximum of social welfare function

$$W^N(U_1, U_2, \cdots, U_n) = U_1 \cdot U_2 \cdot \ldots \cdot U_n \tag{4}$$

in A. We are assuming here that disagreement point gives zero utility to each individual.

As another typical case, we consider the situation where individuals negotiate on the best point in A without knowing what type of utility function and production ability he (or she) will be assigned, although he (or she) knows about all possible types of utility functions. Harsanyi solution (Harsanyi (1955)) $\{(x_i^*(1), y_i^*(1)) \mid i \in N\}$ is defined as the attainable allocation which gives the maximum of

$$W^H(U_1, U_2, \cdots, U_n) = U_1 + U_2 + \cdots + U_n \tag{5}$$

and, Rawls solution (Rawls (1971)) $\{(x_i^*(-\infty), y_i^*(-\infty)), \mid i \in N\}$ is the one which gives the maximum of

$$W^R(U_1, U_2, \cdots, U_n) = \min(U_1, U_2, \cdots, U_n) \,. \tag{6}$$

We refer to Samuelson (1987), Binmore (1989) and Weymark (1991) for further discussions on these concepts.

Each of the above three solutions is obtained as the one which maximizes the social welfare function

$$W(U_1, U_2, \cdots, U_n, \rho) = (U_1^\rho + U_2^\rho + \cdots + U_n^\rho)^{1/\rho} \tag{7}$$

when $\rho \to 0$, $\rho = 1$, $\rho \to -\infty$, respectively. In the sequel we will assume that $\rho \leq 1$. It is clear that maximizing (7) in A is equivalent to maximizing (resp. minimizing)

$$W(U_1, U_2, \cdots, U_n, \rho) = U_1^\rho + U_2^\rho + \cdots + U_n^\rho \tag{8}$$

in the same set for $0 < \rho \leq 1$ (resp. $-\infty < \rho < 0$).

3. FIRST-BEST SOLUTIONS

In this and the next sections, we assume that the government has complete information about the characteristics of individuals and can apply the first-best tax policies. Thus, the government maximizes the social welfare function (7) subject to the budget constraint

$$\sum_i x_i = \sum_i y_i \,. \tag{9}$$

If we write the Lagrangian of the problem as

$$\mathscr{L} = U_1^\rho + U_2^\rho + \cdots + U_n^\rho + \lambda \sum_i (x_i - y_i) \,, \tag{10}$$

then the first best optimal conditions[2] may be expressed as

$$\rho U_i^{\rho-1} v'(x_i) + \lambda = 0 \qquad (i \in N) \tag{11}$$

$$\rho U_i^{\rho-1} c_i'(y_i) + \lambda = 0 \qquad (i \in N) \,. \tag{12}$$

When nothing is raid to the contrary, we will assume that the maximum occurs in the interior of the domain.

Eliminating λ from (11) and (12), we obtain

$$U_i^{\rho-1} v'(x_i) = U_j^{\rho-1} v'(x_j) \qquad (i, j \in N) \tag{13}$$

and

$$v'(x_i) = c_i'(y_i) \qquad (i \in N) \,. \tag{14}$$

From (9), (13) and (14), we can determine x_i and y_i.

When $n = 2$, we can represent Harsanyi and Rawls solutions as in Fig. 1 and 2. The marginal rates of substitution of individual i are given by

$$\frac{dx_i}{dy_i} = \frac{c_i'(y_i)}{v'(x_i)} \qquad (i = 1, 2) \,.$$

Under Assumption 1, the indifference curves are upward sloping and convex downward. We also see that the more able individual has flatter indifference curves at each point by Assumption 1. It should be noted here that we can represent the

[2] We obtain similar first order conditions even when the Lagrangian function is written as

$$\mathscr{L} = (U_1^\rho + U_2^\rho + \cdots + U_n^\rho)^{1/\rho} + \lambda \sum_i (x_i - y_i),$$

of course, for a different Lagrangian multiplier.

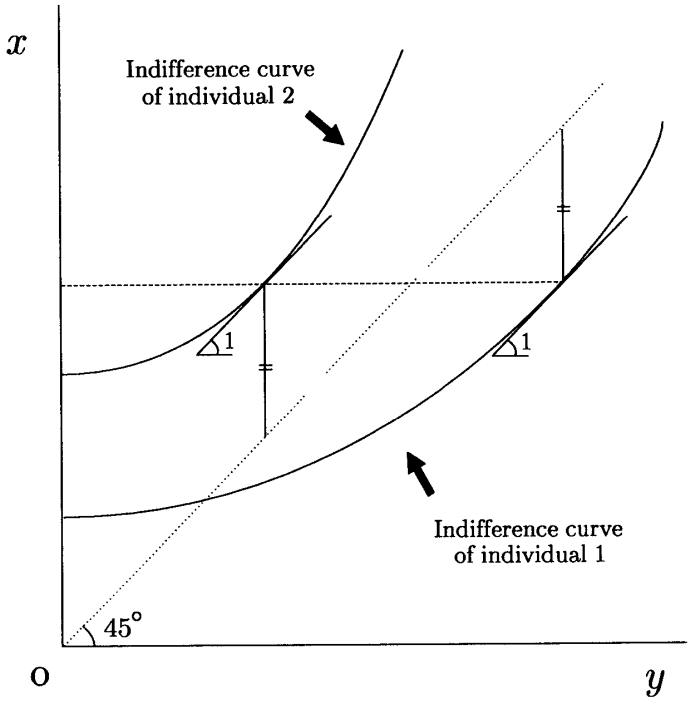

Fig. 1. Harsanyi solution.

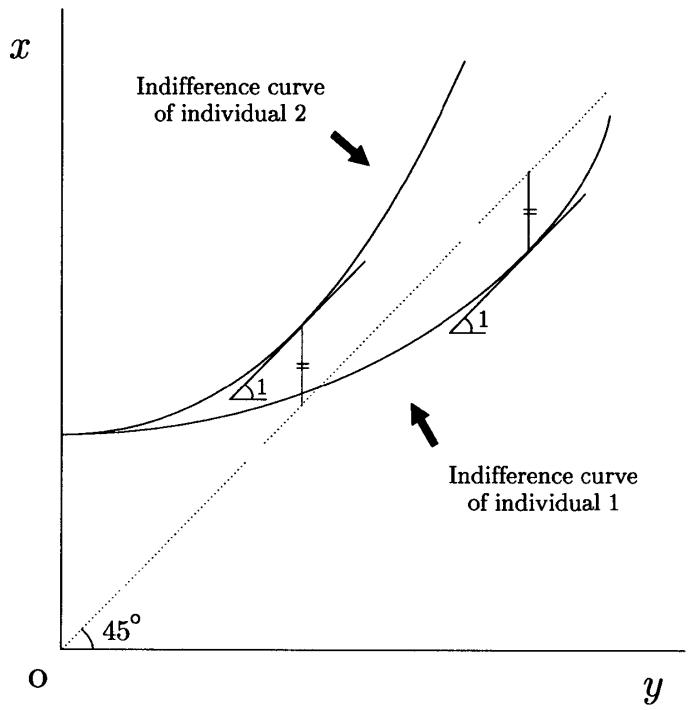

Fig. 2. Rawls solution.

difference of utility levels of the two individuals by the difference of the intercepts of the two indifference curves with the x axis. This follows, since when $y_i=0$ the utility levels can be expressed as $U_i=v(x_i)$ $(i\in N)$ from (1) and Assumption 1.

The budget constraint (9) may be written as $T_1+T_2=0$ and this must be satisfied in both Harsanyi and Rawls solutions. Diagrammatically, the vertical length between the optimal allocation and the 45° line is equal in the absolute value for the two individuals. Also, we see from (14) that, at equilibrium allocations, indifference curves must have unit slope. Finally, we see from (13) that $x_1=x_2$ must hold at the Harsanyi solution and that $U_1=U_2$ must hold at the Rawls solution.

4. COMPARISON OF FIRST-BEST SOLUTIONS WHEN $n=2$

When $n=2$, the budget equation (9) may be written as

$$x_1+x_2=y_1+y_2\,. \tag{15}$$

Differentiating (15) with respect to ρ, we obtain

$$\frac{dx_1}{d\rho}+\frac{dx_2}{d\rho}-\frac{dy_1}{d\rho}-\frac{dy_2}{d\rho}=0\,. \tag{16}$$

Similarly, (14) yields

$$v''(x_i)\frac{dx_i}{d\rho}-c_i''(y_i)\frac{dy_i}{d\rho}=0 \qquad (i=1,2)\,. \tag{17}$$

Also, taking the logarithm of (13) and differentiating with respect to ρ, we have

$$a_1\frac{dx_1}{d\rho}-a_2\frac{dx_2}{d\rho}+b_1\frac{dy_1}{d\rho}-b_2\frac{dy_2}{d\rho}=s \tag{18}$$

where we have set

$$a_i=\frac{(\rho-1)v'(x_i)}{U_i}+\frac{v''(x_i)}{v'(x_i)} \qquad (i=1,2)$$

$$b_i=-\frac{(\rho-1)c_i'(y_i)}{U_i} \qquad (i=1,2)$$

and

$$s=\log U_2-\log U_1\,.$$

For the future reference we note that

$$a_i+b_i=\frac{v''(x_i)}{v'(x_i)} \qquad (i=1,2)\,. \tag{19}$$

In matrix notation, (16), (17), (18) may be written as

$$\begin{pmatrix} v''(x_1) & 0 & -c_1''(y_1) & 0 \\ 0 & v''(x_2) & 0 & -c_2''(y_2) \\ 1 & 1 & -1 & -1 \\ a_1 & -a_2 & b_1 & -b_2 \end{pmatrix} \begin{pmatrix} dx_1/d\rho \\ dx_2/d\rho \\ dy_1/d\rho \\ dy_2/d\rho \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ s \end{pmatrix}. \tag{20}$$

We will denote the determinant of 4×4 matrix of (20) as Δ. Then, in view of (19), we have

$$\Delta = \begin{vmatrix} v''(x_1)-c_1''(y_1) & 0 & -c_1''(y_1) & 0 \\ 0 & v''(x_2)-c_2''(y_2) & 0 & -c_2''(y_2) \\ 0 & 0 & -1 & -1 \\ v''(x_1)/v'(x_1) & -v''(x_2)/v'(x_2) & b_1 & -b_2 \end{vmatrix}.$$

Making use of Assumption 1, we know that

$$\Delta = (v_1''-c_1'')(v_2''-c_2'')(b_1+b_2)+(v_1''-c_1'')\frac{v_2''}{v_2'}c_2''+(v_2''-c_2'')\frac{v_1''}{v_1'}c_1''>0\,,$$

where we have used shorthand notations $v_1' = v'(x_1)$, $v_2' = v'(x_2)$ etc.

From (20), (2) and Assumption 1 we have

$$\frac{dx_1}{d\rho} = \frac{sc_1''(v_2''-c_2'')}{\Delta} < 0$$

$$\frac{dx_2}{d\rho} = -\frac{sc_2''(v_1''-c_1'')}{\Delta} > 0$$

$$\frac{dy_1}{d\rho} = \frac{sv_1''(v_2''-c_2'')}{\Delta} > 0$$

$$\frac{dy_2}{d\rho} = -\frac{sv_2''(v_1''-c_1'')}{\Delta} < 0$$

$$\frac{dT_1}{d\rho} = \frac{s(v_1''-c_1'')(v_2''-c_2'')}{\Delta} > 0$$

$$\frac{dT_2}{d\rho} = -\frac{s(v_1''-c_1'')(v_2''-c_2'')}{\Delta} < 0\,,$$

and, in view of (14), we have

$$\frac{d(T_1/y_1)}{d\rho} = \frac{s(v_2''-c_2'')v_1'}{y_1^2\Delta}\left(\frac{x_1v_1''}{v_1'} - \frac{y_1c_1''}{c_1'}\right) > 0$$

$$\frac{d(T_2/y_2)}{d\rho} = -\frac{s(v_1''-c_1'')v_2'}{y_2^2\Delta}\left(\frac{x_2v_2''}{v_2'} - \frac{y_2c_2''}{c_2'}\right) < 0\,.$$

It will be shown in Appendix 1 that under Assumptions 1 and 2, we have $s>0$.

We now define the elasticity of the disutility function $c_i(y_i)$ (or the degree of the returns to scale of $c_i(y_i)$) as $\varepsilon_i(y_i)=y_i c_i'(y_i)/c_i(y_i)$ and make the following assumption.

ASSUMPTION 2. $\varepsilon_1(y_1)<\varepsilon_2(y_2)$ *for all* $y_1 \geq y_2 > 0$.

The following proposition is proved in Appendix 1.

PROPOSITION 1. *Under Assumptions* 1 *and* 2, *the first best utility levels* U_1 *and* U_2 *are the same for the Rawls social welfare function and* $U_2 > U_1$ *for the Harsanyi and the Nash social welfare functions.*

Hence we may state the following result.

PROPOSITION 2. *Under Assumptions* 1 *and* 2, *the first best amount of tax and the tax rate for the more able* (*resp.*, *less able*) *individual are highest* (*resp.*, *lowest*) *for the Harsanyi social welfare function, second to it for the Nash social welfare function and lowest* (*resp.*, *highest*) *for the Rawls social welfare function. The same* (*resp.*, *opposite*) *order applies for the optimal amount of production for the more able* (*resp.*, *less able*) *individual and the optimal amount of consumption for the less able* (*resp.*, *more able*) *individual.*

5. BEHAVIOR OF AN INDIVIDUAL UNDER A GIVEN TAX FUNCTION AND THE MAXIMIZATION OF SOCIAL WELFARE

In this and the next sections, we assume that a tax function $T(y_i)$ is given for an individual i. Later we will sometimes need to indicate that this function depends on a (vector of) parameter(s) τ_i, in which case we will denote it $T(y_i;\tau_i)$. But for most occasions we delete τ_i and indicate the derivative of the tax function with respect to y_i as $T'(y_i)$, or simply as T_i'.

We make the following assumption on the tax function.

ASSUMPTION 3.

(i) $\dfrac{\partial T(y_i;\tau_i)}{\partial y_i}<1 \qquad (i\in N)$

(ii) $\dfrac{\partial T(y_i;\tau_i)}{\partial \tau_i}>0 \qquad (i\in N)$

The first part of the assumption means that the marginal tax rate is smaller than unity and the second part indicates that τ_i is a parameter that shifts the tax function upward. Typical examples of τ_i include the amount of the lump-sum tax and the proportional tax rate.

We suppose that individual i ($i\in N$) maximizes utility (1) given

$$x_i=y_i-T(y_i)\,. \tag{21}$$

The first order condition $\partial U_i/\partial y_i = 0$ can be written as

$$(1-T'(y_i))v'(x_i) = c_i'(y_i) \qquad (i \in N)\,. \tag{22}$$

When a tax function depends on a parameter τ_i and can be written as $T(y_i;\tau_i)$, (21) yields

$$\frac{dx_i}{d\tau_i} = (1-T'(y_i))\frac{dy_i}{d\tau_i} - T_\tau(y_i)\,,$$

where $T_\tau(y_i)$ denotes the differentiation of $T(y_i;\tau_i)$ with respect to τ_i. In the sequel $T_\tau'(y_i)$ will denote the differentiation of $T'(y_i)$ with respect to τ_i.

Now, if we differentiate

$$U_i = v(y_i - T(y_i;\tau_i)) - c_i(y_i)$$

with respect to τ_i and make use of (22) and Assumption 3, we have

$$\frac{dU_i}{d\tau_i} = \frac{\partial U_i}{\partial y_i}\frac{dy_i}{d\tau_i} - v'(x_i)T_{i\tau} = -v'(x_i)T_{i\tau} < 0\,. \tag{23}$$

Next, taking the logarithm of (22) and differentiating with respect to τ_i, we have

$$\left\{\frac{c_i''(y_i)}{c_i'(y_i)} - \frac{(1-T'(y_i))v''(x_i)}{v'(x_i)} + \frac{T''(y_i)}{(1-T'(y_i))}\right\}\frac{dy_i}{d\tau_i} = -\frac{v''(x_i)}{v'(x_i)}\frac{\partial T_i}{\partial \tau_i} - \frac{T_\tau'(y_i)}{1-T'(y_i)}\,. \tag{24}$$

Now, let us define the degree of relative risk aversion (which reduces to the elasticity of demand if the consumer behaves as a price taker) by

$$\delta_i = -\frac{x_i v''(x_i)}{v'(x_i)}\,.$$

If the tax function is linear and can be written as

$$T(y_i;\tau_i) = \tau_i y_i + \beta_i \quad \text{with } \tau_i, \beta_i \text{ given,}$$

we have $T'(y_i) = \tau_i$, $T''(y_i) = 0$, $T_\tau(y_i) = y_i$, $T_\tau'(y_i) = 1$. Hence (24) becomes

$$\left\{\frac{c_i''(y_i)}{c_i'(y_i)} - \frac{(1-\tau_i)v''(x_i)}{v'(x_i)}\right\}\frac{dy_i}{d\tau_i} = -\frac{v''(x_i)}{v'(x_i)}y_i - \frac{1}{1-\tau_i}\,. \tag{25}$$

In the special case of lump-sum taxation $T = \tau_i$ (i.e. when the above linear tax function can be written as $T(y_i) = \beta_i$, we have $T' = 0$, $T'' = 0$, $T_\tau = 1$, $T_\tau' = 0$. Hence (24) becomes

$$\left\{\frac{c_i''(y_i)}{c_i'(y_i)} - \frac{v''(x_i)}{v'(x_i)}\right\}\frac{dy_i}{d\tau_i} = -\frac{v''(x_i)}{v'(x_i)}\,. \tag{26}$$

Equations (23) and (25) establishes the following:

PROPOSITION 3. *Under Assumptions* 1 *and* 3, *an increase in the shift parameter* τ_i *of the tax function* $T(y_i;\tau_i)$ *decreases the utility of individual i. If, moreover the tax function is linear with* $\beta_i \leq 0$, *then an increase in* τ_i *decreases output* y_i *if* $\delta_i < 1$ *and increases output when* $\beta_i \geq 0$ *and* $\delta_i > 1$.

In the special case of proportional tax, that is when $\beta_i = 0$ in (25), we have the following.

PROPOSITION 4. *Under Assumptions* 1 *and* 3, *an increase in the proportional tax rate* τ_i *increases output* y_i *if* $\delta_i > 1$, *and decreases* y_i *if* $\delta_i < 1$.

Next, we will assume that the government applies separate tax parameter τ_i for each individual so as to maximize the social welfare function (8) given (9).

If we define the Lagrangian function as in (10), the first order condition for maximization with respect to τ_i ($i \in N$), given y_i, yields (11), implying

$$U_i^{\rho-1} v_i' = U_j^{\rho-1} v_j' \qquad (i, j \in N) . \tag{27}$$

This is formally the same as (13), which, in view of (22), is equivalent to

$$\frac{U_i^{\rho-1} c_i'}{1-T_i'} = \frac{U_j^{\rho-1} c_j'}{1-T_j'} \qquad (i, j \in N) . \tag{28}$$

From (9), (21), (22) and (27), all x_i and y_i ($i \in N$) may be determined.

6. COMPARISON OF EQUILIBRIA UNDER A GIVEN TAX RULE

In this section, we assume that $n=2$, in which case the budget constraint may be written as

$$T(y_1;\tau_1) + T(y_2;\tau_2) = 0 . \tag{29}$$

Hence, we have the situation where e.g.,

$$T(y_1) > 0 , \qquad T(y_2) < 0 .$$

Differentiating (21) with respect to parameter ρ in the social welfare function, we have

$$\frac{dx_i}{d\rho} = (1-T_i')\frac{dy_i}{d\rho} - T_{i\tau}\frac{d\tau_i}{d\rho} \qquad (i=1,2) . \tag{30}$$

Next, taking the logarithm of the both sides of (22) and differentiating with respect to ρ, we have

$$-\left\{\frac{T_{i\tau}'}{(1-T_i')T_{i\tau}} + \frac{v''(x_i)}{v'(x_i)}\right\}\frac{dx_i}{d\rho} + \left\{\frac{c_i''(y_i)}{c_i'(y_i)} + \frac{T_i''}{1-T_i'} + \frac{T_{i\tau}'}{T_{i\tau}}\right\}\frac{dy_i}{d\rho} = 0 \qquad (i=1,2) . \tag{31}$$

We now set

$$k_i = -\left\{\frac{T'_{i\tau}}{(1-T'_i)T_{i\tau}} + \frac{v''(x_i)}{v'(x_i)}\right\} \qquad (i=1,2) \tag{32}$$

$$m_i = \frac{c''_i(y_i)}{c'_i(y_i)} + \frac{T''_i}{1-T'_i} + \frac{T'_{i\tau}}{T_{i\tau}} \qquad (i=1,2) \tag{33}$$

and rewrite (31) as

$$k_i \frac{dx_i}{d\rho} + m_i \frac{dy_i}{d\rho} = 0 \qquad (i=1,2)\,. \tag{34}$$

In the case of lump-sum tax, we have $k_i > 0$, $m_i > 0$. Also, in the case of proportional tax, we have

$$k_i = \frac{\delta_i - 1}{x_i}$$

$$m_i = \frac{c''_i}{c'_i} + \frac{1}{y_i} > 0\,,$$

where δ_i is the degree of relative risk aversion. Hence, we know that $k_i > 0$ or $k_i < 0$ depending on whether $\delta_i > 1$ or $\delta_i < 1$. We also know that

$$k_i + m_i = \frac{\delta_i - \tau_i}{x_i} + \frac{c''_i}{c'_i}\,.$$

Here we make the following assumption.

ASSUMPTION 4. *$\delta_i \geq 1$ holds for each individual i ($i \in N$).*

Under Assumption 4, it follows that $k_i \geq 0$ and $k_i + m_i > 0$ for each i ($i \in N$).

The following proposition is proved in Appendix 2.

PROPOSITION 5. *Under Assumptions* 1, 2, 3 *and* 4, *when proportional tax rates are applied, the optimal utility levels U_1 and U_2 are the same for the Rawls social welfare function and $U_2 > U_1$ holds for the Harsanyi and the Nash social welfare functions.*

We will proceed to prove

PROPOSITION 6. *Under Assumptions* 1, 2, 3 *and* 4, *when proportional tax rates are applied, the optimal amount of tax and tax rate for the more able individual (resp., less able individual) are highest (resp., lowest) for the Harsanyi social welfare function, second to it for the Nash social welfare function and lowest (resp., highest) for the Rawls social welfare function. The same (resp., opposite) order applies for the amount of consumption of the less able (resp., more able) individual.*

Proof. Differentiating the budget constraint (15) with respect to ρ, we have

$$\frac{dx_1}{d\rho}+\frac{dx_2}{d\rho}-\frac{dy_1}{d\rho}-\frac{dy_2}{d\rho}=0\,. \tag{35}$$

Also, taking the logarithm of both sides of (27), and differentiating with respect to ρ, we have

$$\left\{\frac{(\rho-1)v'(x_1)}{U_1}+\frac{v''(x_1)}{v'(x_1)}\right\}\frac{dx_1}{d\rho}-\left\{\frac{(\rho-1)v'(x_2)}{U_2}+\frac{v''(x_2)}{v'(x_2)}\right\}\frac{dx_2}{d\rho}$$
$$-\frac{(\rho-1)c_1'(y_1)}{U_1}\frac{dy_1}{d\rho}+\frac{(\rho-1)c_2'(y_2)}{U_2}\frac{dy_2}{d\rho}=s\,. \tag{36}$$

Hence, setting

$$a_i=-\left\{\frac{(\rho-1)v'(x_i)}{U_i}+\frac{v''(x_i)}{v'(x_i)}\right\}\qquad (i=1,2)$$

$$b_i=-\frac{(\rho-1)c_i'(y_i)}{U_i}\qquad (i=1,2)$$

and

$$s=\log U_2-\log U_1\,,$$

we may write (36) as

$$-a_1\frac{dx_1}{d\rho}+a_2\frac{dx_2}{d\rho}+b_1\frac{dy_1}{d\rho}-b_2\frac{dy_2}{d\rho}=s\,. \tag{37}$$

In matrix notation, the above equations (34), (35) and (37) may be expressed as

$$\begin{pmatrix} k_1 & 0 & m_1 & 0\\ 0 & k_2 & 0 & m_2\\ 1 & 1 & -1 & -1\\ -a_1 & a_2 & b_1 & -b_2\end{pmatrix}\begin{pmatrix} dx_1/d\sigma\\ dx_2/d\rho\\ dy_1/d\rho\\ dy_2/d\rho\end{pmatrix}=\begin{pmatrix}0\\0\\0\\s\end{pmatrix}. \tag{38}$$

We denote the determinant of the above 4×4 matrix as Δ. Then a straightforward computation yields

$$\Delta=\begin{vmatrix} k_1+m_1 & 0 & m_1 & 0\\ 0 & k_2+m_2 & 0 & m_2\\ 0 & 0 & -1 & -1\\ b_1-a_1 & a_2-b_2 & b_1 & -b_2\end{vmatrix}$$
$$=(k_1+m_1)(k_2b_2+m_2a_2)+(k_2+m_2)(k_1b_1+m_1a_1)>0\,.$$

The last inequality follows from Assumptions 1, 3 and 4. In Appendix 2, it will be proved that s defined above is positive. From (38) we immediately have

$$\frac{dx_1}{d\rho} = -\frac{sm_1(k_2+m_2)}{\Delta} < 0$$

$$\frac{dx_2}{d\rho} = \frac{sm_2(k_1+m_1)}{\Delta} > 0$$

$$\frac{dy_1}{d\rho} = \frac{sk_1(k_2+m_2)}{\Delta} \geq 0 \tag{39}$$

$$\frac{dy_2}{d\rho} = -\frac{sk_2(k_1+m_1)}{\Delta} \leq 0 .$$

From these equations, we also have

$$\frac{dT_1}{d\rho} = \frac{s(k_1+m_1)(k_2+m_2)}{\Delta} > 0$$

$$\frac{dT_2}{d\rho} = -\frac{s(k_1+m_1)(k_2+m_2)}{\Delta} < 0$$

$$\frac{d(T_1/y_1)}{d\rho} = \frac{s(k_2+m_2)(x_1k_1+y_1m_1)}{y_1^2\Delta} > 0 \tag{40}$$

$$\frac{d(T_2/y_2)}{d\rho} = -\frac{s(k_1+m_1)(x_2k_2+y_2m_2)}{y_2^2\Delta} < 0 . \quad Q.E.D.$$

7. ASYMMETRIC INFORMATION

In this section, following Stiglitz (1982, 1987), we consider the situation where the government has limited information about the abilities of agents. We recall that individual 1 (type 1) is more productive than individual 2 (type 2), i.e. individual 1 feels less disutility than individual 2 to obtain the same output. We are assuming here that the government cannot distinguish who is type 1 and who is type 2. Therefore, the government must care about whether individual 1 (resp., 2) would pretend to be type 2 (resp., 1) after it has chosen tax rules. This additional requirement at the equilibrium allocation can be expressed as

$$U_1(x_1, y_1) \geq U_1(x_2, y_2) \tag{41}$$

$$U_2(x_2, y_2) \geq U_2(x_1, y_1) , \tag{42}$$

which will be referred to as the incentive constraints. Here we examine whether the optimal solutions (x_i, y_i) $(i=1, 2)$ obtained in Section 3 satisfy the incentive constraints. The Harsanyi solution is represented in Fig. 1 and the Rawls solution, in Fig. 2. We can see from Figures 1 and 2 that the incentive constraint (41) is not satisfied at the optimal allocations.

This can be proved rigorously as follows. From (1) we obtain

$$U_1(x_1, y_1) - U_1(x_2, y_2) = v(x_1) - c_1(y_1) - (v(x_2) - c_1(y_2)) . \tag{43}$$

At the Rawls solution where utility levels of two individuals are the same, we see that the utility difference in (43) can be evaluated as

$$\begin{aligned} &v(x_1) - c_1(y_1) - (v(x_2) - c_2(y_2)) + c_1(y_2) - c_2(y_2) \\ &= c_1(y_2) - c_2(y_2) < 0 . \end{aligned}$$

Differentiating (43) with respect to ρ, we obtain

$$v_1' \frac{dx_1}{d\rho} - c_1' \frac{dy_1}{d\rho} - v_2' \frac{dx_2}{d\rho} + c_1' \frac{dy_2}{d\rho}$$

which is negative under the assumptions of Proposition 2. Therefore, we have

$$U_1(x_1, y_1) < U_1(x_2, y_2)$$

at both Harsanyi and Nash solutions.

Similarly, we can see that the other incentive constraint (42) holds at Harsanyi, Nash and Rawls solutions with strict inequality.

Consequently, we may state the following.

PROPOSITION 7. *Under Assumptions* 1 *and* 2, *the more able individual has an incentive to pretend to be the less able at the first best Harsanyi, Nash and Rawls solutions, while the less able individual does not have an incentive to pretend to be the more able.*

Under given tax functions of Sections 5 and 6, we can evaluate the derivative of the utility difference in (43) as

$$\begin{aligned} \frac{d(U_1(x_1, y_1) - U_1(x_2, y_2))}{d\rho} &= v_1' \frac{dx_1}{d\rho} - c_1' \frac{dy_1}{d\rho} - v_2' \frac{dx_2}{d\rho} + c_1' \frac{dy_2}{d\rho} \\ &= -s\{(v_1' m_1 + c_1' k_1)(k_2 + m_2) + (v_2' m_2 + c_2' k_2)(k_1 + m_1)\}/\Delta , \end{aligned}$$

using (39). We know that, under Assumptions 1, 3 and 4, $v_i' m_i + c_i' k_i > 0$, $k_i + m_i > 0$, $\Delta > 0$. Hence we have the following proposition.

PROPOSITION 8. *Under Assumptions* 1, 2, 3 *and* 4, *given separate proportional tax functions, the more able individual has an incentive to pretend to be the less able at the Harsanyi, Nash and Rawls solutions, while the less able individual does not have an incentive to pretend to be the more able.*

8. CONCLUSION

We have compared the optimal outputs, consumptions and tax rates corresponding to Harsanyi, Nash and Rawls solutions. Some positive propositions were established when there are two types of consumers, and specific studies were

made on proportional taxation. However, the extension for the case of more than two types of consumers remains open. We hope to leave this problem for future study.

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APPENDIX 1: UTILITY FRONTIER AND THE FIRST BEST EQUILIBRIUM UTILITIES

Given the budget constraint

$$x_1 + x_2 = y_1 + y_2 \tag{A.1}$$

a pair of utilities (U_1, U_2) is attainable, if $U_i = v(x_i) - c_i(y_i)$ for some (x_i, y_i) $(i = 1, 2)$ satisfying (9). The utility possibility set is defined as the set of all possible attainable utility pairs and its north-east frontier is referred to as the utility frontier. It is clear from the concavity of the utility functions (1) and the linearity of the budget constraint (9) that utility possibility set is convex. Let μ be the utility level of the consumer 1 and define the Lagrangian

$$\mathscr{L} = U_2 + \alpha(U_1 - \mu) + \lambda(x_1 + x_2 - y_1 - y_2)\,, \tag{A.2}$$

where α and μ are Lagrangian multipliers.

Differentiating $\mathscr{L}$ with respect to x_1, x_2, y_1 and y_2, we obtain the first order conditions for optimality:

$$\alpha v'(x_1) + \lambda = 0$$

$$v'(x_2) + \lambda = 0$$

$$\alpha c_1'(y_1) + \lambda = 0$$

$$c_2'(y_2)+\lambda=0\,.$$

In view of these conditions we also obtain

$$\frac{d\mathscr{L}}{d\mu}=\frac{dU_2}{dU_1}=-\alpha \tag{A.3}$$

$$\lambda=-c_2'(y_2)=-v'(x_2) \tag{A.4}$$

$$v'(x_i)-c_i'(y_i)=0 \qquad (i=1,2) \tag{A.5}$$

and

$$\alpha=\frac{v'(x_2)}{v'(x_1)}\,. \tag{A.6}$$

From (A.3) and (A.6), the slope of the utility frontier is given as $\alpha=v_2'/v_1'$. We will now establish Proposition 1 in Section 4.

Proof of Proposition 1. When the slope of the utility frontier is unity, then

$$v'(x_1)=v'(x_2)\,,$$

hence we have $x_1=x_2$. From (A.5) we then have

$$c_1'(y_1)=c_2'(y_2)\,.$$

Now suppose that $y_1\leq y_2$. Then Assumption 1 (ii) implies

$$c_1'(y_1)\leq c_1'(y_2)<c_2'(y_2)$$

contradicting the above equality. Hence we must have $y_1>y_2$.

When the slope of the utility frontier is unity, we saw that $x_1=x_2$, hence using (1) and (A.5) we obtain

$$\begin{aligned}\frac{U_2-U_1}{v_1'(x_1)}&=\frac{c_1(y_1)}{c_1'(y_1)}-\frac{c_2(y_2)}{c_2'(y_2)}\\&=\frac{y_1}{\varepsilon_1(y_1)}-\frac{y_2}{\varepsilon_2(y_2)}\,,\end{aligned}$$

where $\varepsilon_i=y_ic_i'(y_i)/c_i(y_i)$. Hence Assumption 2 implies $U_2>U_1$. This proves the proposition for the Harsanyi solution.

The proposition is obvious for the Rawls solution. As to the Nash solution, the line connecting the origin and the equilibrium point must have equal but opposite slope as the tangent line to the utility frontier (see Nash (1953)). Hence it cannot lie below the 45° line since the utility frontier is convex upward and its slope is unity when $x_1=x_2$. *Q.E.D.*

APPENDIX 2: UTILITY FRONTIER UNDER A GIVEN TAX RULE

In this Appendix, we will give conditions under which the utility possibility set is convex in the economy with two individuals where each individual maximizes his or her utility under a given tax function. Formally we will prove

LEMMA 1. *Under Assumptions* 1, 2, 3 *and* 4, *in the economy with two individuals, the utility frontier is downward sloping and convex upward when each individual maximizes his or her utility under a given tax function* $T(y_i; \tau_i)$.

Proof. We define the Lagrangian

$$\mathscr{L} = U_2 + \alpha(U_1 - \mu) + \lambda(T(y_1; \tau_1) + T(y_2; \tau_2)) \tag{A.7}$$

in the same way as (A.2). Differentiating $\mathscr{L}$ with respect to τ_i $(i=1, 2)$, we obtain the following first order conditions:

$$-\alpha v'(x_1)T_{1\tau} + \lambda T_{1\tau} = 0$$

$$-v'(x_2)T_{2\tau} + \lambda T_{2\tau} = 0 .$$

Also we have

$$\frac{d\mathscr{L}}{d\mu} = \frac{dU_2}{dU_1} = -\alpha . \tag{A.8}$$

From the first order conditions, we obtain

$$\alpha = \frac{v'(x_2)}{v'(x_1)} . \tag{A.9}$$

Next, taking the logarithm of (22) and differentiating with respect to μ, we have

$$k_i \frac{dx_i}{d\mu} + m_i \frac{dy_i}{d\mu} = 0 \qquad (i=1, 2) \tag{A.10}$$

where k_i and m_i are defined in (32) and (33) respectively. Differentiating (15) with respect to μ we have

$$\frac{dx_1}{d\mu} + \frac{dx_2}{d\mu} - \frac{dy_1}{d\mu} - \frac{dy_2}{d\mu} = 0 . \tag{A.11}$$

Also, differentiating $U_1 = \mu$ i.e.

$$v(x_1) - c_1(y_1) = \mu$$

with respect to μ, we obtain

$$v'(x_1)\frac{dx_1}{d\mu} - c_1'(y_1)\frac{dy_1}{d\mu} = 1 . \tag{A.12}$$

In matrix notation, (A.10), (A.11), (A.12) may be summarized as

$$\begin{pmatrix} k_1 & 0 & m_1 & 0 \\ 0 & k_2 & 0 & m_2 \\ 1 & 1 & -1 & -1 \\ v'(x_1) & 0 & -c_1'(y_1) & 0 \end{pmatrix} \begin{pmatrix} dx_1/d\mu \\ dx_2/d\mu \\ dy_1/d\mu \\ dy_2/d\mu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{A.13}$$

Let us denote the determinant of 4×4 matrix of (A.13) as Δ. Δ can be calculated as

$$\Delta = \begin{vmatrix} k_1+m_1 & 0 & m_1 & 0 \\ 0 & k_2+m_2 & 0 & m_2 \\ 0 & 0 & -1 & -1 \\ v_1'-c_1' & 0 & -c_1' & 0 \end{vmatrix}$$
$$= -(k_2+m_2)(v'(x_1)m_1+c_1'(y_1)k_1) < 0 .$$

In fact, making use of Assumption 4, we know that $k_2+m_2>0$. In the case of lump-sum taxation, due to the fact that k_i, $m_i>0$, we obtain $\Delta<0$. In the case of proportional taxation, we can derive

$$v'(x_1)m_1+c_1'(y_1)k_1 = v_1'\left(\frac{c_1''}{c_1'}+\frac{\delta_1}{y_1}\right)>0$$

which ensures that $\Delta<0$. From (A.13), Assumption 1 and Assumption 4 we have

$$\begin{aligned} \frac{dx_1}{d\mu} &= -\frac{m_1(k_2+m_2)}{\Delta} > 0 \\ \frac{dx_2}{d\mu} &= \frac{m_2(k_1+m_1)}{\Delta} < 0 \\ \frac{dy_1}{d\mu} &= \frac{k_1(k_2+m_2)}{\Delta} \\ \frac{dy_2}{d\mu} &= -\frac{k_2(k_1+m_1)}{\Delta} . \end{aligned} \tag{A.14}$$

Also taking the logarithm of $\alpha = v_2'/v_1'$ and differentiating with respect to μ, we have

$$\frac{1}{\alpha}\frac{d\alpha}{d\mu} = \frac{v_2''}{v_2'}\frac{dx_2}{d\mu} - \frac{v_1''}{v_1'}\frac{dx_1}{d\mu} > 0 .$$

Therefore, we obtain

$$-\frac{dU_2}{dU_1} = \alpha > 0$$

and

$$-\frac{d^2U_2}{dU_1^2}=\frac{d\alpha}{d\mu}>0\,,$$

which proves the lemma. *Q.E.D.*

We will now establish Proposition 5 in Section 6.

Poof of Proposition 5. In view of (A.8) and (A.9), the slope of the utility frontier is given by $\alpha=v_2'/v_1'$. It is equal to unity when

$$v'(x_1)=v'(x_2)$$

that is, when $x_1=x_2$. Using $v_1'=v_2'$ and (22) we have

$$(1-T'(y_2))c_1'(y_1)=(1-T'(y_1))c_2'(y_2)\,. \tag{A.15}$$

Here we assume that marginal tax rates are non-decreasing, i.e. if $y_1\leq y_2$, then $T'(y_1)\leq T'(y_2)$. With this assumption, using (A.15) and Assumption 1, we can derive $y_1>y_2$.

In the case where proportional tax is imposed, (A.15) can be written as

$$(1-\tau_2)c_1'(y_1)=(1-\tau_1)c_2'(y_2)\,. \tag{A.16}$$

Also, due to the fact that $x_1=x_2$, we have

$$(1-\tau_1)y_1=(1-\tau_2)y_2\,. \tag{A.17}$$

Dividing (A.16) by (A.17) we have

$$c_1'(y_1)y_1=c_2'(y_2)y_2\,. \tag{A.18}$$

From (1) and (A.18) it follows that

$$\begin{aligned}U_2-U_1&=c_1(y_1)-c_2(y_2)\\&=c_1'(y_1)y_1\left[\frac{1}{\varepsilon_1(y_1)}-\frac{1}{\varepsilon_2(y_2)}\right].\end{aligned}$$

By Assumption 2, we have $U_2>U_1$. This proves the proposition for the Harsanyi solution. The proposition is obvious for the Rawls social welfare function.

As to Nash solution, the line connecting the origin and the equilibrium point must have equal but opposite slope as the tangent line to the utility frontier at the point. Hence it cannot lie below the 45° line since the utility frontier is convex upward and its slope is unity when $x_1=x_2$. *Q.E.D.*