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Another Proof of Ostrowski-Kolchin-Hardouin Theorem in Difference Algebra
差分代数におけるOstrowski-Kolchin-Hardouin 定理の別証明

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This paper gives another proof of an analog of Ostrowski-Kolchin theorem in difference algebra, which was proved by Hardouin. Let K be a field of characteristic 0 and \((L, \tau)\) a difference extension of a difference field \((K, \tau)\). Denote the invariant field of \((L, \tau)\) and that of \((K, \tau)\) by \(C_L\) and \(C\) respectively. Suppose \(C\) is an algebraically closed field. Suppose \(x_1, \ldots, x_m, y_1, \ldots, y_n\) are nonzero elements of \(L\) satisfying \(\tau(x_i) = u_i x_i\) and \(\tau(y_j) = y_j + v_j\), where \(u_i, \ldots, u_m, v_1, \ldots, v_n \in K\). Then the analog states that if \(x_1, \ldots, x_m, y_1, \ldots, y_n\) are algebraically dependent over \(KC\), there exists a nonzero element \(a \in K\) satisfying \(\tau(a) = (\prod_{i=1}^m u_i^{k_i}) a\) for a nonzero element \((k_i) \in \mathbb{Z}^m\) or \(\tau(a) = a + \sum_{j=1}^n a_j v_j\) for a nonzero element \((a_j) \in C^n\).

本論文では Hardouin による Ostrowski-Kolchin の定理の差分化の別証明を与えた。\(K\) を標数 0 の体、\((L, \tau)\) を \((K, \tau)\) の差分拡大とする。\((L, \tau)\) と \((K, \tau)\) の不变体をそれぞれ \(C_L\) と表記し、\(C\) は代数閉体とする。\(L\) の 0 でない元 \(x_1, \ldots, x_m, y_1, \ldots, y_n\) が \(u_i, \ldots, u_m, v_1, \ldots, v_n \in K\) に対して \(\tau(x_i) = u_i x_i, \tau(y_j) = y_j + v_j\) を満たすと仮定する。このとき差分化された定理は次のように記述される：\(x_1, \ldots, x_m, y_1, \ldots, y_n\) が \(KC\) 上代数的従属ならば、ある 0 でない元 \(a \in K\) が存在して \(\tau(a) = (\prod_{i=1}^m u_i^{k_i}) a\) となる 0 でない \((k_i) \in \mathbb{Z}^m\) が存在する、または \(\tau(a) = a + \sum_{j=1}^n a_j v_j\) となる 0 でない \((a_j) \in C^n\) が存在する。

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1 Introduction

In [1], Hardouin has proved an analog of Ostrowski-Kolchin theorem with one derivation operator [2] using difference Galois theory. The purpose of this paper is to give another proof in difference algebra, in which we use module of differentials and its fundamental propositions instead.

To state our theorem we prepare some notions in difference algebra (cf. [4, pp.103 -115]). We always regard any ring (field) as a commutative ring (field) with characteristic 0. Let \( K \) be a field and \( \tau \) an isomorphism from \( K \) to itself. We call the pair \((K, \tau)\) a difference field and \( \tau \) the transforming operator of \( K \).

Let \( L \) be an extension of \( K \) which is also a difference field with a transforming operator \( \tau_L \). We call \((L, \tau_L)\) a difference extension of \((K, \tau)\) if \( \tau_L|_K = \tau_K \). By \( C_K \), we denote the invariant field of \((K, \tau)\), that is, the field of invariant elements of \( \tau_K \). For \( a = (a_1, \ldots, a_n) \in K^n \) and \( k = (k_1, \ldots, k_n) \in \mathbb{Z}^n \), we put \( a^k = \prod_{i=1}^{n} a_i^{k_i} \).

Then we shall show the following theorem:

**Theorem** Let \((L, \tau)\) be a difference extension of a difference field \((K, \tau)\) and the invariant field \( C = C_K \) an algebraically closed field. Suppose \( x_1, \ldots, x_m, y_1, \ldots, y_n \) are nonzero elements of \( L \) satisfying

\[
\tau(x_i) = u_i x_i, \quad \tau(y_j) = y_j + v_j,
\]

where \( u_1, \ldots, u_m, v_1, \ldots, v_n \in K \). If \( x_1, \ldots, x_m, y_1, \ldots, y_n \) are algebraically dependent over \( KC_\tau \), then there exists a nonzero element \( a \in K \) satisfying

\[
\tau(a) = \left( \prod_{i=1}^{m} u_i^{k_i} \right) a
\]

for a nonzero element \( (k) \in \mathbb{Z}^m \) or

\[
\tau(a) = a + \sum_{i=1}^{n} a_i y_i
\]

for a nonzero element \( (a) \in C^\tau \).

2 Preliminaries

Let \( A \) be an algebra over a ring \( R \). There exist an \( A \)-module \( \Omega \) called the module of differentials of \( A \) over \( R \) and an \( R \)-linear derivation \( d : A \rightarrow \Omega \) called the universal \( R \)-linear derivation if for any \( A \)-module \( M \) and any \( R \)-linear derivation \( D : A \rightarrow M \) there is a unique \( A \)-module homomorphism \( f : \Omega \rightarrow M \) such that \( D = f \circ d \) (cf. [4, pp.91-92]). The following propositions are well-known:

**Proposition 1** (Rosenlicht [6]). Let \( L/K \) be a field extension and \( \Omega \) its module of differentials with the universal \( K \)-linear derivation \( d \). Then \( \eta_1, \ldots, \eta_s \in L \) are algebraically independent over \( K \) if and only if \( d\eta_1, \ldots, d\eta_s \in \Omega \) are linearly independent over \( L \).

**Proposition 2** (Rosenlicht [6]). Let \( L/K \) be a field extension and \( \Omega \) its module of differentials with the universal \( K \)-linear derivation \( d \). Suppose \( a_1, \ldots, a_n \in K \) are linearly independent over \( \mathbb{Q} \). If \( \eta_1, \ldots, \eta_s \in L \) satisfy

\[
\sum_{i=1}^{n} a_i \frac{d\eta_i}{\eta_i} = 0,
\]

then \( d\eta = d\eta_1 \cdots d\eta_s = 0 \).

**Proposition 3** (Kubota [3]). Suppose \((L, \tau)\) is a difference extension of a difference field \((K, \tau)\). Let \( \Omega \) be the module of differentials of \( L/K \) with the universal \( K \)-linear derivation \( d \). Then there exists an additive mapping \( \tau^* : \Omega \rightarrow \Omega \) such that

\[
\tau^*(\eta d\xi) = \tau(\eta)d(\tau(\xi)) \quad (\eta, \xi \in L).
\]

3 Proof of Theorem

From the assumption, we may suppose that \( L \) is finitely generated over \( K \). In fact, there is a nonzero polynomial \( F \in KC_\tau[X_1, \ldots, X_m, Y_1, \ldots, Y_n] \) satisfying

\[
F(x_1, \ldots, x_m, y_1, \ldots, y_n) = 0.
\]
Let $L'$ be an extension over $K$ generated by $x_1, \ldots, x_m, y_1, \ldots, y_n$ and the elements of $C_L$ being in the coefficients of $F$. Then we have $\tau (L') \subset L'$, so that $(L', \tau )/(K, \tau )$ is a difference extension. Furthermore, we only have to prove our theorem in case $y_1, \ldots, y_n$ are algebraically independent over $C$.

First, suppose that $x_1, \ldots, x_m$ are algebraically dependent over $K C_L$ and take the minimal number $m'$ such that $x_1, \ldots, x_{m'}$ are algebraically dependent over $K C_L$. Let $\Omega$ be the module of differentials of $L/K C_L$ with the universal $K C_L$-linear derivation $d : L \to \Omega$. Then there is a nontrivial equation of linear dependence over $L$,

$$\sum_{i=1}^{m'} a_i \frac{dx_i}{x_i} = 0,$$

where $a_i \in L$ and $a_m = 1$. Applying the additive mapping $\tau^*$ of Proposition 3 to this equation, we have

$$\sum_{i=1}^{m'} (\tau (a_i)) \frac{dx_i}{x_i} = 0.$$

Hence we get by $\tau (a_m) = a_m = 1$,

$$\sum_{i=1}^{m-1} (\tau (a_i)) \frac{dx_i}{x_i} = 0.$$

Since $dx_1, \ldots, dx_{m-1}$ are linearly independent over $L$ from Proposition 1, it follows that $\tau (a_i) = a_i$ for each $i$. Hence every $a_i$ is a member of $C_L$. There are elements $c_1, \ldots, c_r \in C_L$ such that they are linearly independent over $\mathbb{Q}$ and satisfy

$$a_i = \sum_{j=1}^{r} h_j c_j (h_j \in \mathbb{Z}).$$

Then not all $h_j$ are zero. Putting $z_i = \prod_{j=1}^{r} x_i^{h_j}$, we have

$$\sum_{j=1}^{r} c_j \frac{dz_j}{z_j} = \sum_{i=1}^{m'} a_i \frac{dx_i}{x_i} = \sum_{i=1}^{m'} a_i \frac{dx_i}{x_i} = 0.$$

From Proposition 2, each $z_i$ is algebraic over $K C_L$. Take some $z_i$ of them such that not all $a_{i_1}, \ldots, a_{i_n}$ are zero. Considering its minimal polynomial, we can take a nonzero element $z \in K C_L$ satisfying

$$\tau (z) = u^{(r, n)} z,$$

where $r$ is a positive integer and $(r, n) = (r, n_1, \ldots, r_n, n_m)$.

Next, suppose that $x_1, \ldots, x_m$ are algebraically independent over $K C_L$. Take the minimal number $n'$ such that $x_1, \ldots, x_{m'}, y_1, \ldots, y_{n'}$ are algebraically dependent over $K C_L$. There is a nontrivial equation of linear dependence over $L$,

$$\sum_{i=1}^{m'} a_i \frac{dx_i}{x_i} + \sum_{k=1}^{n'} b_k dy_k = 0,$$

where $a_i, b_k \in L$ and $b_m = 1$. Applying $\tau^*$ to this equation, we have

$$\sum_{i=1}^{m'} (\tau (a_i)) \frac{dx_i}{x_i} + \sum_{k=1}^{n'} (\tau (b_k)) dy_k = 0.$$

Hence $a_i, b_k$ are included in $C_L$. Take $c_1, \ldots, c_r \in C_L$, $n_k \in \mathbb{Z}$ and $z \in L$ following the same procedure as above. Then we get

$$\sum_{j=1}^{r} c_j \frac{dz_j}{z_j} + d \left( \sum_{k=1}^{n'} b_k y_k \right) = 0.$$

Hence $\sum_{j=1}^{r} c_j y_k$ is algebraic over $K C_L$. There is also an element $w \in K C_L$ satisfying

$$\tau (w) = w - \sum_{k=1}^{n'} r_k b_k y_k$$

for some positive integer $r_k$.

We can embed $C_L$ into the field of formal power
series $C((t))$ over $C$ as a field, since $C_i$ is finitely generated over $C$. We see $C_i$ and $K$ are linearly disjoint over $C$, and so are $K$ and $C((t))$. In fact, suppose $a_1, \ldots, a_r \in C_i$ are linearly dependent over $K$. If $r = 1$, clearly $a_1$ is linearly dependent over $C$. Assume that $a_1, \ldots, a_r \in C_i$ are linearly dependent over $K$. There are $k_1, \ldots, k_r \in K$ with $k_r = 1$ such that

$$\sum_{i=1}^r k_i a_i = 0.$$ 

Applying $\tau$ to this, we have

$$\sum_{i=1}^r \tau(k_i) a_i = 0.$$ 

Hence we obtain $\tau(k_i) = k_i$, so that $k_i \in C$. This means $a_1, \ldots, a_r$ are linearly dependent over $C$. Next, suppose $k_1, \ldots, k_r \in K$ are linearly dependent over $C((t))$. Then there are formal power series $\sum_{r=0}^\infty c_r t^r, \ldots, \sum_{r=0}^\infty c_r t^r \in C((t))$ which make a nontrivial equation of linear dependence,

$$\sum_{r=1}^\infty k_r c_r t^r = \sum_{r=1}^\infty \left( \sum_{r'=1}^\infty k_r c_{r'} \right) t^r = 0.$$ 

So we get $\sum_{r=1}^\infty k_r c_{r_v} = 0$ for all $v$. Since some $c_v$ is a nonzero element, $k_1, \ldots, k_r$ are linearly dependent over $C$.

Hence there is an embedding from $KC_i$ into the field of formal power series $K((t))$ over $K$ as a difference field defining $\tau$ on $K((t))$. The above $z$ can be described in $K((t))$ as

$$z = \sum_{r=1}^\infty a_r t^r \quad (a_r \in K, \quad a_r \neq 0).$$

From (3), we see

$$\sum_{r=0}^\infty \tau(a_r) t^r = \sum_{r=0}^\infty a_r \sum_{r'=0}^\infty b_{r,r'} t^{r'}.$$ 

Therefore we obtain $\tau(a_r) = a_r \sum_{r'=0}^\infty b_{r,r'} t^{r'}$, the form of (1).

We also put

$$w = \sum_{r=0}^\infty \gamma_r t^r, \quad b_k = \sum_{r=0}^\infty \beta_{k,r} t^r,$$

where $\gamma_r \in K, \beta_{k,r} \in C$ and $\beta_{k,0} = 1$. From (4), we see

$$\sum_{r=0}^\infty \tau(\gamma_r) t^r = \sum_{r=0}^\infty \gamma_r t^r - \sum_{r=1}^\infty r_a v_beta_{r} t^r = \sum_{r=0}^\infty \left( \gamma_r - \sum_{r'=0}^\infty \beta_{r,r'} v_r \right) t^r.$$ 

Since $v_1, \ldots, v_{n}$ are linearly independent over $C$ from the assumption, we see $\gamma_0 \neq 0$. Therefore $\gamma_0$ is a nonzero element satisfying $\tau(\gamma_0) = \gamma_0 - \sum_{r=0}^\infty r_a v_beta_{r} t^r$, the form of (2).

References


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